Notes on p-adic Hodge theory

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Contents

CHAPTER I

Introduction

1. A first glimpse of p-adic Hodge theory

Our goal in this section is to give a rough idea of what p-adic Hodge theory is about. By nature, p-adic Hodge theory has two sides of the story, namely the arithmetic side and the geometric side. We will briefly motivate and describe each side of the story, and discuss how the two sides are related.

1.1. The arithmetic perspective

From the arithmetic perspective, p-adic Hodge theory is the study of p-adic Galois representations. i.e., continuous representations $\Gamma_K := \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}_n(\mathbb{Q}_p)$ where K is a p-adic field. This turns out to be much more subtle and interesting than the study of ℓ-adic Galois representations, i.e. continuous representations $\Gamma_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ with $\ell \neq p$. In the ℓ-adic case, the topologies on Γ_K and $\mathbb{Q}_\ell$ do not get along with each other very well, thereby imposing a huge restriction on the kinds of continuous representations that we can have. In the p-adic case, on the other hand, we don't have this "clash" between the topologies on Γ_K and $\mathbb{Q}_p$, and consequently have much more Galois representations than in the ℓ-adic case.

Remark. Our definition of p-adic field allows infinite extensions of $\mathbb{Q}_p$. For a precise definition, see Definition 3.1.1 in Chapter II.

In this subsection, we discuss a toy example to motivate and demonstrate some key ideas from the arithmetic side of p-adic Hodge theory. Let E be an elliptic curve over $\mathbb{Q}_p$ with good reduction. This means that we have a unique elliptic scheme $\mathcal{E}$ over $\mathbb{Z}_p$ with $\mathcal{E}_{\mathbb{Q}_p} \simeq E$. For each prime ℓ (which may be equal to p), the Tate module

$$T_\ell(E) := \varprojlim E[\ell^n](\overline{\mathbb{Q}}_p) \simeq \mathbb{Z}_\ell^2$$

is equipped with a continuous $G_{\mathbb{Q}_p}$-action, which means that the rational Tate module

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^2$$

is an ℓ-adic Galois representation. The Tate module $T_\ell(E)$ and the rational Tate module $V_\ell(E)$ contain important information about E, as suggested by the following fact:

Fact. For two elliptic curves E_1 and E_2 over $\mathbb{Q}_p$, the maps

$$\begin{aligned} \mathrm{Hom}(E_1, E_2) \otimes \mathbb{Z}_\ell &\longrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}_p}}(T_\ell(E_1), T_\ell(E_2)) \\ \mathrm{Hom}(E_1, E_2) \otimes \mathbb{Q}_\ell &\longrightarrow \mathrm{Hom}_{\Gamma_{\mathbb{Q}_p}}(V_\ell(E_1), V_\ell(E_2)) \end{aligned} \tag{1.1}$$

are injective; in other words, a map between E_1 and E_2 is determined by the induced map on the (rational) Tate modules as Galois representations.

Remark. The above fact remains true if $\mathbb{Q}_p$ is replaced by an arbitrary field L of characteristic is not equal to ℓ. Moreover, the maps in (1.1) become isomorphism if L is a finite field, a global function field or a number field, as shown respectively by Tate, Zarhin and Faltings.

For $\ell \neq p$, we can explicitly describe the Galois action on $T_\ell(E)$ (and $V_\ell(E)$) by passing to the mod p reduction $\mathcal{E}_{\mathbb{F}_p}$ of $\mathcal{E}$. Note that $\mathcal{E}_{\mathbb{F}_p}$ is an elliptic curve over a finite field $\mathbb{F}_p$, so the Galois action of $\Gamma_{\mathbb{F}_p}$ on the Tate module $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ and the rational Tate module $V_\ell(\mathcal{E}_{\mathbb{F}_p})$ are very well understood. In fact, the Frobenius element of $\Gamma_{\mathbb{F}_p}$, which topologically generates the Galois group $\Gamma_{\mathbb{F}_p}$, acts on $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ with characteristic polynomial $x^2 - ax + p$ where $a = p + 1 - \#\mathcal{E}_{\mathbb{F}_p}(\mathbb{F}_p)$. Now the punchline is that we have isomorphisms

$$T_\ell(E) \simeq T_\ell(\mathcal{E}_{\mathbb{F}_p}) \qquad \text{and} \qquad V_\ell(E) \simeq V_\ell(\mathcal{E}_{\mathbb{F}_p}) \tag{1.2}$$

as $\Gamma_{\mathbb{Q}_p}$-representations, where the actions on $T_\ell(\mathcal{E}_{\mathbb{F}_p})$ and $V_\ell(\mathcal{E}_{\mathbb{F}_p})$ are given by $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$. In other words, we can describe the Galois action on $T_\ell(E)$ (and $V_\ell(E)$) as follows:

(1) The action of $\Gamma_{\mathbb{Q}_p}$ factors through the map $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$.
(2) The Frobenius element of $G_{\mathbb{F}_p}$ acts with characteristic polynomial $x^2 - ax + p$ where $a = p + 1 - \#\mathcal{E}_{\mathbb{F}_p}(\mathbb{F}_p)$.

A Galois representation of $\Gamma_{\mathbb{Q}_p}$ which satisfies the statement (1) is said to be *unramified*. The terminology comes from the fact that $\Gamma_{\mathbb{F}_p}$ is isomorphic to $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$, where $\mathbb{Q}_p^{\mathrm{un}}$ denotes the maximal unramified extension of $\mathbb{Q}_p$. It is worthwhile to mention that our discussion in the preceding paragraph shows one direction of the following important criterion:

Theorem 1.1.1 (Néron [**Nér64**], Ogg [**Ogg67**], Shafarevich)**.** *An elliptic curve E on $\mathbb{Q}_p$ has a good reduction if and only if the Tate module $T_\ell(E)$ is unramified for all primes $\ell \neq p$.*

Let us now turn to the case $\ell = p$, which is our primary interest. In this case, we never have an isomorphism between the (rational) Tate modules as in (1.2); indeed, $T_p(\mathcal{E}_{\mathbb{F}_p})$ is isomorphic to either $\mathbb{Z}_p$ or 0 whereas $T_p(E)$ is always isomorphic to $\mathbb{Z}_p^2$. This suggests that the action of $\Gamma_{\mathbb{Q}_p}$ on $T_p(E)$ has a nontrivial contribution from the kernel of the map $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$, called the *inertia group* of $\mathbb{Q}_p$, which we denote by $I_{\mathbb{Q}_p}$.

Therefore we need another invariant of E which does not lose too much information about the Galois action under passage to the mod p reduction $\mathcal{E}_{\mathbb{F}_p}$. A solution by Grothendieck and Tate is to replace the Tate module $T_p(E)$ by the direct limit of p-power torsion groups

$$E[p^\infty] := \varinjlim E[p^n]$$

called the *p-divisible group* of E. Here we consider each $E[p^n]$ as a finite flat group scheme over $\mathbb{Q}_p$. It is not hard to see that $E[p^\infty]$ contains all information about the Galois action on $T_p(E)$ in the following sense:

Fact. We can recover the Galois action of $\Gamma_{\mathbb{Q}_p}$ on $T_p(E)$ from $E[p^\infty]$.

We can similarly define the p-divisible groups $\mathcal{E}[p^\infty]$ and $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$ associated to $\mathcal{E}$ and $\mathcal{E}_{\mathbb{F}_p}$. The p-divisible groups of $E, \mathcal{E}$ and $\mathcal{E}_{\mathbb{F}_p}$ are related as follows:

$$\begin{array}{ccc} & \mathcal{E}[p^\infty] & \\ {\scriptstyle \otimes\mathbb{Q}_p} \swarrow & & \searrow {\scriptstyle \otimes\mathbb{F}_p} \\ E[p^\infty] & & \mathcal{E}_{\mathbb{F}_p}[p^\infty] \end{array}$$

We wish to study $E[p^\infty]$ using $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$, as we expect that the theory of p-divisible groups becomes simpler over $\mathbb{F}_p$ than it is over $\mathbb{Q}_p$. The first step towards this end is provided by the following fundamental result:

Theorem 1.1.2 (Tate [**Tat67**]). *The generic fiber functor*

$$\left\{ \begin{array}{c} p\text{-divisible groups} \\ \text{over } \mathbb{Z}_p \end{array} \right\} \xrightarrow{\otimes \mathbb{Q}_p} \left\{ \begin{array}{c} p\text{-divisible groups} \\ \text{over } \mathbb{Q}_p \end{array} \right\}$$

is fully faithful.

Remark. Theorem 1.1.2 is the main result of Tate's seminal paper [**Tat67**], which marks the true beginning of p-adic Hodge theory. Here we already see how this result provides the first significant progress in the arithmetic side of the theory. In the next subsection we will see how its proof initiates the geometric side of the theory.

Let us now consider the problem of studying $\mathcal{E}[p^\infty]$ using the mod p reduction $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$. Here the key is to realize $\mathcal{E}[p^\infty]$ as a characteristic 0 lift of $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$. More precisely, we identify the category of p-divisible groups over $\mathbb{Z}_p$ with the category of p-divisible groups over $\mathbb{F}_p$ equipped with "lifting data". Such an identification is obtained by switching to another category, as stated in the following fundamental result:

Theorem 1.1.3 (Dieudonné [**Die55**], Fontaine [**Fon77**]). *There are (anti-)equivalences of categories*

$$\left\{ \begin{array}{c} p\text{-divisible groups} \\ \text{over } \mathbb{F}_p \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Dieudonné modules} \\ \text{over } \mathbb{F}_p \end{array} \right\}$$
$$\left\{ \begin{array}{c} p\text{-divisible groups} \\ \text{over } \mathbb{Z}_p \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Dieudonné modules over } \mathbb{F}_p \\ \text{with an ``admissible'' filtration} \end{array} \right\}$$

where a Dieudonné module over $\mathbb{F}_p$ means a finite free $\mathbb{Z}_p$-module M equipped with a (Frobenius-semilinear) endomorphism φ such that $pM \subset \varphi(M)$.

Remark. The description of Dieudonné modules in our situation is misleadingly simple. In general, the endomorphism φ should be Frobenius-semilinear in an appropriate sense. Here we don't get this semilinearity since the Frobenius automorphism of $\mathbb{F}_p$ acts trivially on $\mathbb{Z}_p$.

We have thus transformed the study of the p-adic Galois action on the Tate modules to the study of certain explicit semilinear algebraic objects. Roughly speaking, the actions of the inertia group $I_{\mathbb{Q}_p}$ and the Frobenius element in $\Gamma_{\mathbb{F}_p}$ on $T_p(E)$ are respectively encoded by the "admissible" filtration and the (semilinear) endomorphism φ on the corresponding Dieudonné module.

If we instead want to study the p-adic Galois representation of the rational Tate module, all we have to do is to invert p in the corresponding Dieudonné module. The resulting algebraic object is a finite dimensional vector space over $\mathbb{Q}_p$ with a (Frobenius-semilinear) automorphism φ. Such an object is called an *isocrystal*.

Our discussion here shows an example of the defining theme of p-adic Hodge theory. In fact, much of p-adic Hodge theory is about constructing a dictionary that relates good categories of p-adic Galois representations to various categories of semilinear algebraic objects. The dictionary that we described here serves as a prototype for many other dictionaries.

Another recurring theme of p-adic Hodge theory is base change of the ground field K to the completion $\widehat{K^{\text{un}}}$ of its maximal unramified extension. In terms of the residue field, this amounts to passing to the algebraic closure. In most cases, such a base change preserves key information about the Galois action of Γ_K. In fact, most good properties of p-adic representations of Γ_K turn out to be detected on the inertia group, which is preserved under passing to $\widehat{K^{\text{un}}}$ as follows:

$$I_K \simeq \Gamma_{K^{\text{un}}} \simeq \Gamma_{\widehat{K^{\text{un}}}}.$$

Moreover, base change to $\widehat{K^{\mathrm{un}}}$ often greatly simplifies the study of the Galois action of Γ_K. For example, in our discussion base change to $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ amounts to replacing the residue field by $\overline{\mathbb{F}}_p$, thereby allowing us to make use of the following fundamental result:

Theorem 1.1.4 (Manin [**Man63**])**.** *The category of isocrystals over $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ is semisimple.*

In summary, we have motivated and described several key ideas in p-adic Hodge theory via Galois representations that arise from an elliptic curve over $\mathbb{Q}_p$ with good reduction. In particular, our discussion shows a couple of recurring themes in p-adic Hodge theory, as stated below.

(1) Construction of a dictionary between good categories of p-adic representations and various categories of semilinear algebraic objects.
(2) Base change of the ground field K to $\widehat{K^{\mathrm{un}}}$.

It is natural to ask whether there is a general framework for these themes. To answer this question, we need to investigate the geometric side of the story.

1.2. The geometric perspective

From the geometric perspective, p-adic Hodge theory is the study of the geometry of a (proper smooth) variety X over a p-adic field K. Our particular interests are various cohomology theories related to X, such as

- the étale cohomology $H^n_{\text{ét}}$,
- the algebraic de Rham cohomology H^n_{dR},
- the crystalline cohomology H^n_{cris}.

Note that p-adic Galois representations naturally come into play via the étale cohomology groups $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Hence we already see a vague connection to the arithmetic side of p-adic Hodge theory.

In this subsection, we motivate and state three fundamental comparison theorems about these cohomology theories. These theorems share a general theme of extracting some information about the geometry of X from the Γ_K-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$.

Recall that, for a proper smooth $\mathbb{C}$-scheme Y, we have the *Hodge decomposition*

$$H^n(Y(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{i+j=n} H^i(Y, \Omega^j_Y).$$

During the proof of Theorem 1.1.2, Tate observed the existence of an analogous decomposition for the étale cohomology of an abelian variety over K with good reduction. This discovery led to his conjecture that such a decomposition should exist for all étale cohomology groups of an arbitrary proper smooth varieties over K. This conjecture is now a theorem, commonly referred to as the *Hodge-Tate decomposition*.

Theorem 1.2.1 (Faltings [**Fal88**])**.** *Let $\mathbb{C}_K$ denote the p-adic completion of $\overline{K}$. For a proper smooth variety X over K, there is a canonical isomorphism*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_K(-j) \tag{1.3}$$

compatible with Γ_K-actions.

Remark. Since the action of Γ_K on $\overline{K}$ is continuous, it uniquely extends to an action on $\mathbb{C}_K$. Thus Γ_K acts diagonally on the left side of (1.3) and only through the Tate twists $\mathbb{C}_K(-j)$ on the right side of (1.3).

For an analogy to other two comparison theorems, let us rewrite (1.3) as

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}} \cong \Big(\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \Big) \otimes_K B_{\text{HT}}$$

where $B_{\text{HT}} := \underset{j \in \mathbb{Z}}{\oplus} \mathbb{C}_K(j)$ is the *Hodge-Tate period ring.* By a theorem of Tate and Sen, we have $B_{\text{HT}}^{\Gamma_K} = K$. Hence we obtain an isomorphism of finite dimensional graded K-vector spaces

$$\big(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}}\big)^{\Gamma_K} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}),$$

which allows us to recover the Hodge numbers from the Γ_K-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$.

Next we discuss the comparison theorem between étale cohomology and de Rham cohomology. Recall that, for a proper smooth $\mathbb{C}$-scheme Y of dimension d, we have a comparison isomorphism

$$H^n(Y(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^n_{\text{dR}}(Y/\mathbb{C}) \tag{1.4}$$

given by Poincare duality and the "period paring"

$$H^n_{dR}(Y(\mathbb{C})/\mathbb{C}) \times H_{2d-n}(Y(\mathbb{C}), \mathbb{C}) \longrightarrow \mathbb{C}, \qquad (\omega, \Gamma) \mapsto \int_\Gamma \omega.$$

One may hope to obtain a p-adic analogue of (1.4) by tensoring both $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ and $H^n_{\text{dR}}(X/K)$ with an appropriate "period ring". Fontaine [**Fon82**] formulated this idea into a conjecture using his construction of a ring B_{dR} that satisfies the following properties:

(1) B_{dR} is equipped with a filtration such that the associated graded ring is B_{HT}.

(2) B_{dR} is endowed with an action of Γ_K such that $B_{\text{dR}}^{\Gamma_K} = K$.

Below is a precise statement of this conjecture, which is now a theorem commonly referred to as the *p-adic de Rham comparison isomorphism.*

Theorem 1.2.2 (Faltings [**Fal88**]). *For a proper smooth variety X over K, there is a canonical isomorphism*

$$H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H^n_{\text{dR}}(X/K) \otimes_K B_{\text{dR}} \tag{1.5}$$

compatible with Γ_K-actions and filtrations.

Remark. By construction, the de Rham cohomology group $H^n_{\text{dR}}(X/K)$ is endowed with the *Hodge filtration* whose associated graded K-space is the Hodge cohomology $\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$. The filtration on the right side of (1.5) is given by the convolution filtration.

An important consequence of Theorem 1.2.2 is that one can recover the de Rham cohomology $H^n_{\text{dR}}(X/K)$ from the Γ_K-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ by

$$\big(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}}\big)^{\Gamma_K} \cong H^n_{\text{dR}}(X/K).$$

Moreover, one can recover Theorem 1.2.1 from Theorem 1.2.2 by passing to the associated graded K-vector spaces.

However, Theorem 1.2.2 (or Theorem 1.2.1) does not provide any way to recover the Γ_K-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. It is therefore natural to seek for a refinement of $H^n_{\text{dR}}(X/K)$ which recovers the Γ_K-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. Grothendieck conjectured that, when X has good reduction, such a refinement should be given by the crystalline cohomology in the following sense:

Conjecture 1.2.3 (Grothendieck [**Gro74**]). *Let k be the residue field of $\mathcal{O}_K$. Denote by $W(k)$ the ring of Witt vectors over k, and by K_0 its fraction field. There should exist a (purely algebraic) fully faithful functor $\mathcal{D}$ on a category of certain p-adic Galois representations such that*

$$\mathcal{D}\big(H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)\big) = H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes_{W(k)} K_0$$

for all proper smooth variety X with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$.

Remark. The functor $\mathcal{D}$ in Conjecture 1.2.3 has become to known as the *Grothendieck mysterious functor.*

It is worthwhile to mention that Conjecture 1.2.3 is motivated by the dictionary that we described in 1.1. Recall that, for an elliptic curve E over $\mathbb{Q}_p$ with good reduction, we discussed how the $\Gamma_{\mathbb{Q}_p}$-representation on $V_p(E)$ is determined by the associated filtered isocrystal. We may regard this dictionary as a special case of the Grothendieck mysterious functor, as $V_p(E)$ and the associated filtered isocrystal are respectively identified with the dual of $H^1_{\text{ét}}(E_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$ and $H^1_{\mathrm{cris}}(\mathcal{E}_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The key insight of Grothendieck was that there should be a way to go directly from $H^1_{\text{ét}}(E_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$ to $H^1_{\mathrm{cris}}(\mathcal{E}_{\mathbb{F}_p}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ without using p-divisible groups.

Fontaine [**Fon94**] reformulated Conjecture 1.2.3 in terms of a comparison isomorphism between étale cohomology and crystalline cohomology. His idea is to construct another period ring B_{cris} that satisfies the following properties:

(1) B_{cris} is equipped with an action of Γ_K such that $B_{\mathrm{cris}}^{\Gamma_K} = K_0$.
(2) There is a Frobenius-semilinear endomorphism φ on B_{cris}.
(3) There is a natural map

$$B_{\mathrm{cris}} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}$$

which induces a filtration on B_{cris} from the filtration on B_{dR}.

The endomorphism φ in (2) is referred to as the Frobenius action on B_{cris}. Fontaine's conjecture is now a theorem, which we state as follows:

Theorem 1.2.4 (Faltings [**Fal88**]). *Suppose that X has good reduction, meaning that it has a proper smooth model $\mathcal{X}$ over $\mathcal{O}_K$. There exists a canonical isomorphism*

$$H^n_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] \otimes_{K_0} B_{\mathrm{cris}}$$

compatible with Γ_K-actions, filtrations, and Frobenius actions.

Remark. By construction, the crystalline cohomology $H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ carries a natural Frobenius action. Moreover, the Hodge filtration on $H^n_{\mathrm{dR}}(X/K)$ induces a filtration on $H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ via the comparison isomorphism

$$H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p] \otimes_{K_0} K \cong H^n_{\mathrm{dR}}(X/K).$$

By Theorem 1.2.4, we have an isomorphism

$$\big(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}\big)^{\Gamma_K} \cong H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p].$$

With some additional work, we can further show that the Γ_K-action on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ can be recovered from $H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ by taking the filtration and the Frobenius action into account. In fact, the mysterious functor in Conjecture 1.2.3 turns out to be

$$\mathcal{D}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}.$$

However, we still need to specify the source and the target categories such that $\mathcal{D}$ is fully faithful. The answer turns out to come from the interplay between the arithmetic and the geometric perspectives, as we will see in the next subsection.

1.3. The interplay via representation theory

The Grothendieck mysterious functor, which we have yet to give a complete description, is an example of various functors that link the arithmetic side and the geometric side of p-adic Hodge theory. Such functors provide vital means for studying p-adic Hodge theory via the interplay between the arithmetic and geometric perspectives.

Here we describe a general formalism due to Fontaine for constructing functors that connect the arithmetic and geometric sides of p-adic Hodge theory. Let $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ denote the category of p-adic representations of Γ_K for a p-adic field K. For a p-adic period ring B, such as B_{HT}, B_{dR} or B_{cris} as introduced in the preceding subsection, we define

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \qquad \text{for each } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K).$$

We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if the natural morphism

$$\alpha_V : D_B(V) \otimes_{B^{\Gamma_K}} B \longrightarrow V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism. Let $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ be the full subcategory of B-admissible representations. Then D_B defines a functor from $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ to the category of finite dimensional vector spaces over B^{Γ_K} with some additional structures. Here the additional structures that we consider for the target category reflect the structure of the ring B, as indicated by the following examples:

(a) The target category of $D_{B_{\mathrm{HT}}}$ is the category of finite dimensional graded K-spaces, reflecting the graded algebra structure on B_{HT}.
(b) The target category of $D_{B_{\mathrm{dR}}}$ is the category of finite dimensional filtered K-spaces, reflecting the filtration on B_{dR}.
(c) The target category of $D_{B_{\mathrm{cris}}}$ is the category of finite dimensional filtered K_0-spaces with a Frobenius-semilinear endomorphism, reflecting the filtration and the Frobenius action on B_{cris}.

In particular, we have a complete description of the Grothendieck mysterious functor given by $D_{B_{\mathrm{cris}}}$. We also obtain its fully faithfulness from the following fundamental result:

Theorem 1.3.1 (Fontaine [**Fon94**]). *The functors $D_{B_{\mathrm{HT}}}$, $D_{B_{\mathrm{dR}}}$, and $D_{B_{\mathrm{cris}}}$ are all exact and faithful. Moreover, the functor $D_{B_{\mathrm{cris}}}$ is fully faithful.*

Remark. We will see in Chapter III that the first statement of Theorem 1.3.1 is (almost) a formal consequence of some algebraic properties shared by $B_{\mathrm{HT}}, B_{\mathrm{dR}}$ and B_{cris}.

Note that, for each $B = B_{\mathrm{HT}}, B_{\mathrm{dR}},$ or B_{cris}, the definition of B-admissibility is motivated by the corresponding comparison theorem from the preceding subsection, while the target category of the functor D_B consists of semilinear algebraic objects that arise in the arithmetic side of p-adic Hodge theory. In other words, the functor D_B relates a certain class of "geometric" p-adic representations to a class of semilinear algebraic objects that carry some arithmetic information. Hence we can consider Fontaine's formalism as a general framework for connecting the following themes:

(1) Study of the geometry of a proper smooth variety over a p-adic field via the Galois action on the étale cohomology groups.
(2) Construction of a dictionary that relates certain p-adic representations to various semilinear algebraic objects.

In fact, this tidy connection provided by Fontaine's formalism forms the backbone of classical p-adic Hodge theory.

2. A first glimpse of the Fargues-Fontaine curve

In this section, we provide a brief introduction to a remarkable geometric object called the *Fargues-Fontaine curve*, which serves as the fundamental curve of p-adic Hodge theory. Our goal for this section is twofold: building some intuition about what this object is, and explaining why this object plays a pivotal role in modern p-adic Hodge theory.

2.1. Definition and some key features

The Fargues-Fontaine curve has two different incarnations, namely the schematic curve and the adic curve. In this subsection, we will only consider the schematic curve, as we don't have a language to describe the adic curve. The two incarnations are essentially equivalent due to a GAGA type theorem, as we will see in Chapter IV.

Throughout this section, let us restrict our attention to the case $K = \mathbb{Q}_p$ for simplicity. We denote by F the completion of the algebraic closure of $\mathbb{F}_p((u))$. Recall that Fontaine constructed a p-adic period ring B_{cris} which is equipped with a $\Gamma_{\mathbb{Q}_p}$-action and a Frobenius semilinear endomorphism φ. There is also a subring B_{cris}^+ of B_{cris} with the following properties:

(i) B_{cris}^+ is stable under φ with $(B_{\text{cris}}^+)^{\varphi=1} \simeq \mathbb{Q}_p$.
(ii) there is an element $t \in B_{\text{cris}}^+$ with $B_{\text{cris}} = B_{\text{cris}}^+[1/t]$ and $\varphi(t) = pt$.

Definition 2.1.1. The *schematic Fargues-Fontaine curve* (associated to the pair $(\mathbb{Q}_p, F)$) is defined by

$$X := \text{Proj}\left(\bigoplus_{n\geq 0}(B_{\text{cris}}^+)^{\varphi=p^n}\right).$$

Note that X can be regarded as a $\mathbb{Q}_p$-scheme by the property (i) of B_{cris}^+. However, as we will see in a moment, the scheme X is not of finite type over $\mathbb{Q}_p$. In particular, X is not a curve in the usual sense, and not even a projective scheme over $\mathbb{Q}_p$.

Nonetheless, the scheme X is not completely exotic. In fact, X is geometrically akin to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ in many aspects.

Theorem 2.1.2 (Fargues-Fontaine [**FF18**])**.** *We have the following facts about the Fargues-Fontaine curve X:*

(1) As a $\mathbb{Q}_p$-scheme, X is noetherian, connected and regular of dimension 1*.*
(2) X is a union of two spectra of Dedekind domains.
(3) X is complete in the sense that the divisor of every rational function on X has degree zero.
(4) $\text{Pic}(X) \simeq \mathbb{Z}$*.*

Remark. The statements (1) and (3) together suggest that X behaves almost as a proper curve, thereby justifying the use of the word "curve" to describe X.

We can also describe X as an affine scheme of a principal domain plus "a point at infinity", in the same way as we describe $\mathbb{P}^1_{\mathbb{C}}$ as $\text{Spec}\,(\mathbb{C}[z])$ plus a point at infinity. More precisely, for some "preferred" closed point $\infty \in X$ we have identifications

$$X - \{\infty\} = \text{Spec}\,(B_e) \qquad \text{and} \qquad \widehat{\mathcal{O}_{X,\infty}} = B_{\text{dR}}^+$$

where $B_e := B_{\text{cris}}^{\varphi=1}$ and B_{dR}^+ is the ring of integers of B_{dR}. The fact that B_e is a principal ideal domain is due to Fontaine.

Remark. The above discussion provides a geometric description of the period ring B_{dR}.

2.2. Relation to the theory of perfectoid spaces

The Fargues-Fontaine curve turns out to have a surprising connection to Scholze's theory of *perfectoid spaces*. In this subsection, we describe this connection after recalling some basic definitions and fundamental facts about perfectoid fields.

Definition 2.2.1. Let C be a nonarchimedean field of residue characteristic p.

(1) C is called a *perfectoid field* if it satisfies the following conditions:
   (i) its valuation is nondiscrete,
   (ii) the p-th power Frobenius map on $\mathcal{O}_C/p$ is surjective.

(2) If C is a perfectoid field with valuation $|\cdot|$, we define the *tilt* of C by

$$C^\flat := \varprojlim_{x \mapsto x^p} C$$

which carries a ring structure with a valuation $|\cdot|^\flat$ as follows:
   (a) $(a \cdot b)_n := a_n b_n$,
   (b) $(a + b)_n := \lim\limits_{m \to \infty} (a_{m+n} + b_{m+n})^{p^m}$,
   (c) $|a|^\flat := |a_0|$.

Remark. It is not hard to see that $C^\flat$ is a perfectoid field of characteristic p.

Example 2.2.2. The p-adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ is a perfectoid field with $\mathbb{C}_p^\flat \simeq F$.

The theory of perfectoid fields (and perfectoid spaces) has numerous applications in p-adic Hodge theory. As a key example, we mention Scholze's generalization of Theorem 1.2.2 (and Theorem 1.2.1) to the category of *rigid analytic varieties*. Here we state one of the fundamental results for such applications, known as the *tilting equivalence*.

Theorem 2.2.3 (Scholze [**Sch12**]). *Let C be a perfectoid space.*

(1) Every finite extension of C is a perfectoid field.

(2) The tilting operation induces an equivalence of categories

$$\{ \textit{finite extensions of } C \} \stackrel{\sim}{\longleftrightarrow} \left\{ \textit{finite extensions of } C^\flat \right\}.$$

(3) There is an isomorphism $\Gamma_C \simeq \Gamma_{C^\flat}$ of absolute Galois groups.

An amazing fact is that, gievn a characteristic p perfectoid field F, the Fargues-Fontaine curve X parametrizes the characteristic 0 *untilts* of F, which are pairs (C, ι) consisting of a characteristic 0 perfectoid field C and an isomorphism $\iota : C^\flat \simeq F$. Note that there is an obvious notion of isomorphism for untilts of F. In addition, the p-th power Frobenius automorphism φ_F of F acts on untilts of F by $\varphi_F \cdot (C, \iota) := (C, \varphi_F \circ \iota)$.

Theorem 2.2.4 (Fargues-Fontaine [**FF18**]). *For every closed point $x \in X$, the residue field $k(x)$ is a perfectoid field of characteristic 0 with $k(x)^\flat \simeq F$. Moreover, there is a bijection*

$$\{ \textit{closed points of } X \} \stackrel{\sim}{\longleftrightarrow} \{ \varphi_F\textit{-orbits of characteristic 0 untilts of } F \}$$

given by $x \mapsto \big(k(x), k(x)^\flat \simeq F\big)$.

Remark. Theorem 2.2.4 implies that X is not of finite type over $\mathbb{Q}_p$.

This moduli interpretation of the Fargues-Fontaine curve is one of the main inspirations for Scholze's theory of *diamonds*, which is a perfectoid analogue of Artin's theory of algebraic spaces. In fact, many perfectoid spaces or diamonds that arise in p-adic geometry have moduli interpretations involving (vector bundles on) the Fargues-Fontaine curve.

2.3. Geometrization of p-adic Galois representations

Let us now demonstrate how the Fargues-Fontaine curve provides a way to geometrically study p-adic Galois representations. The geometric objects that we will consider are as follows:

Definition 2.3.1. Let us fix a closed point $\infty \in X$.

(1) A *vector bundle* on X is a locally free $\mathcal{O}_X$-module of finite rank.
(2) A *modification of vector bundles at* ∞ is a tuple $(\mathcal{E}, \mathcal{F}, i)$ where
- $\mathcal{E}$ and $\mathcal{F}$ are vector bundles on X,
- $i : \mathcal{E}|_{X-\{\infty\}} \xrightarrow{\sim} \mathcal{F}|_{X-\{\infty\}}$ is an isomorphism outside ∞.

We will see in Chapter IV that vector bundles on the Fargues-Fontaine curve admit a complete classification. The following theorem summarizes some of its key consequences.

Theorem 2.3.2 (Fargues-Fontaine [**FF18**])**.** *There is a functorial commutative diagram*

$$\begin{array}{ccc} \{\text{ isocrystals over } \overline{\mathbb{F}}_p \} & \xrightarrow{\sim} & \{\text{ vector bundles on } X \} \\ \uparrow & & \uparrow \\ \left\{ \begin{array}{c} \text{filtered isocrystals} \\ \text{over } \overline{\mathbb{F}}_p \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{modifications of} \\ \text{vector bundles on } X \end{array} \right\} \end{array}$$

where the vertical maps are forgetful maps defined by $(N, \mathrm{Fil}^\bullet(N)) \mapsto N$ *and* $(\mathcal{E}, \mathcal{F}, i) \mapsto \mathcal{E}$.

Now recall that Fontaine constructed a fully faithful functor

$$D_{B_{\mathrm{cris}}} : \left\{ \begin{array}{c} B_{\mathrm{cris}}\text{-admissible } p\text{-adic} \\ \text{representations of } \Gamma_{\mathbb{Q}_p} \end{array} \right\} \longrightarrow \{\text{ filtered isocrystals over } \mathbb{F}_p \}.$$

If we compose $D_{B_{\mathrm{cris}}}$ with the base change functor to $\overline{\mathbb{F}}_p$ and the bottom map in Theorem 2.3.2, we obtain a functor

$$\left\{ \begin{array}{c} B_{\mathrm{cris}}\text{-admissible } p\text{-adic} \\ \text{representations of } \Gamma_{\mathbb{Q}_p} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{modifications of} \\ \text{vector bundles on } X \end{array} \right\}.$$

Hence we can study B_{cris}-admissible p-adic Galois representations by purely geometric objects, namely modifications of vector bundles on X. As an application, we obtain the following fundamental result:

Theorem 2.3.3 (Colmez-Fontaine [**CF00**])**.** *Let* $N^\bullet := \big(N, \mathrm{Fil}^\bullet(N)\big)$ *be a filtered isocrystal over* $\mathbb{F}_p$*. We denote by* $\overline{N}^\bullet := \big((\overline{N}), \mathrm{Fil}^\bullet(\overline{N})\big)$ *the associated filtered isocrystal over* $\overline{\mathbb{F}}_p$ *(obtained by base change), and by* $\big(\mathcal{E}(\overline{N}^\bullet), \mathcal{F}(\overline{N}^\bullet), i(\overline{N}^\bullet)\big)$ *its image under the bottom map in Theorem 2.3.2. Then* $N^\bullet$ *is in the essential image of* $D_{B_{\mathrm{cris}}}$ *if and only if the vector bundle* $\mathcal{F}(\overline{N}^\bullet)$ *is trivial (i.e., isomorphic to* $\mathcal{O}_X^{\oplus n}$ *for some* n*).*

Remark. Since $D_{B_{\mathrm{cris}}}$ is fully faithful, its essential image gives a purely algebraic category which is equivalent to the category of B_{cris}-admissible representations.

Theorem 2.3.3 is commonly stated as "weakly admissible filtered isocrystals are admissible". It was initially proved by Colmez-Fontaine [**CF00**] through a very complicated and technical argument. In Chapter IV, we will provide a very short and conceptual proof of Theorem 2.3.3. The key point of our proof is that the left inverse $V_{B_{\mathrm{cris}}}$ of $D_{B_{\mathrm{cris}}}$ can be cohomologically realized by the following identity:

$$V_{B_{\mathrm{cris}}}(N^\bullet) \simeq H^0\big(X, \mathcal{F}(\overline{N}^\bullet)\big).$$

Theorem 2.3.3 has a couple of interesting implications as follows:

(1) B_{cris}-admissibility is a "geometric" property.
(2) B_{cris}-admissibility is insensitive to replacing the residue field $\mathbb{F}_p$ by $\overline{\mathbb{F}}_p$, which amounts to replacing the ground field $\mathbb{Q}_p$ by $\widehat{\mathbb{Q}_p^{\text{un}}}$.

These two implications are closely related since base change of the ground field $\mathbb{Q}_p$ to $\widehat{\mathbb{Q}_p^{\text{un}}}$ can be regarded as "passing to the geometry" via the bottom map in Theorem 2.3.2.

Remark. The Fargues-Fontaine curve also provides a way to geometrically study ℓ-adic Galois representations. In fact, Fargues [**Far16**] initiated a remarkable problem called *the geometrization of the local Langlands correspondence*, which aims to realize the local Langlands correspondence as the geometric Langlands correspondence on the Fargues-Fontaine curve.

CHAPTER II

Foundations of p-adic Hodge theory

1. Finite flat group schemes

In this section we develop some basic theory of finite flat group schemes, in preparation for our discussion of p-divisible groups in §2. Our primary reference is Tate's article [**Tat97**].

Throughout this section, all rings are assumed to be commutative.

1.1. Basic definitions and properties

We begin by recalling the notion of group scheme.

Definition 1.1.1. Let S be a scheme. A *group scheme* over S is an S-scheme G along with morphisms

- $m : G \times_S G \longrightarrow G$, called the *multiplication*,
- $e : S \longrightarrow G$, called the *unit section*,
- $i : G \longrightarrow G$, called the *inverse*,

that fit into the following commutative diagrams:

(a) associativity axiom:

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{(m,\mathrm{id})} & G \times_S G \\ \Big\downarrow{\scriptstyle (\mathrm{id},m)} & & \Big\downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow{\quad m \quad} & G \end{array}$$

(b) identity axiom:

$$\begin{array}{ccccc} G \times_S S & \xrightarrow{\sim} & G & \xrightarrow{\mathrm{id}} & G \\ & {\scriptstyle (\mathrm{id},e)} \searrow & & \nearrow {\scriptstyle m} & \\ & & G \times_S G & & \end{array} \qquad \begin{array}{ccccc} S \times_S G & \xrightarrow{\sim} & G & \xrightarrow{\mathrm{id}} & G \\ & {\scriptstyle (e,\mathrm{id})} \searrow & & \nearrow {\scriptstyle m} & \\ & & G \times_S G & & \end{array}$$

(c) inverse axiom:

$$\begin{array}{ccc} G & \overset{(i,\mathrm{id})}{\underset{(\mathrm{id},i)}{\rightrightarrows}} & G \times_S G \\ \Big\downarrow & & \Big\downarrow{\scriptstyle m} \\ S & \xrightarrow{\quad e \quad} & G \end{array}$$

In other words, a group scheme over S is a group object in the category of S-schemes.

Lemma 1.1.2. *Given a scheme S, an S-scheme G is a group scheme if and only if the set $G(T)$ for any S-scheme T carries a functorial group structure.*

Proof. This is immediate by Yoneda's lemma. □

Definition 1.1.3. Let G and H be group schemes over a scheme S.

(1) A morphism $G \to H$ of S-schemes is called a *homomorphism* if for any S-scheme T the induced map $G(T) \to H(T)$ is a group homomorphism.
(2) The *kernel* of a homomorphism $f : G \to H$, denoted by $\ker(f)$, is a group scheme such that $\ker(f)(T)$ for any S-scheme T is the kernel of the induced map $G(T) \to H(T)$. Equivalently, by Lemma 1.1.2 $\ker(f)$ is the fiber of f over the unit section of H.

Example 1.1.4. Let G be a group scheme over a scheme S, and let n be a positive integer. The *multiplication by* n on G, denoted by $[n]_G$, is a homomorphism $G \to G$ defined by $g \mapsto g^n$.

In this section, we are mostly interested in affine group schemes over an affine base. Let us generally denote the base ring by R.

Definition 1.1.5. Let $G = \mathrm{Spec}\,(A)$ be an affine group scheme over R. We define

- the *comultiplication* $\mu : A \to A \otimes_R A$,
- the *counit* $\epsilon : A \to R$
- the *coinverse* $\iota : A \to A$,

to be the maps respectively induced by the multiplication, unit section, and inverse of G.

Example 1.1.6. We present some important examples of affine group schemes.

(1) The *additive group over* R is a scheme $\mathbb{G}_a := \mathrm{Spec}\,(R[t])$ with the natural additive group structure on $\mathbb{G}_a(B) = B$ for each R-algebra B. The comultiplication, counit, and coinverse are given by
$$\mu(t) = t \otimes 1 + 1 \otimes t, \qquad \epsilon(t) = 0, \qquad \iota(t) = -t.$$
(2) The *multiplicative group over* R is a scheme $\mathbb{G}_m := \mathrm{Spec}\,(R[t, t^{-1}])$ with the natural multiplicative group structure on $\mathbb{G}_m(B) = B^\times$ for each R-algebra B. The comultiplication, counit, and coinverse are given by
$$\mu(t) = t \otimes t, \qquad \epsilon(t) = 1, \qquad \iota(t) = t^{-1}.$$
(3) The *n-th roots of unity* is a scheme $\mu_n := \mathrm{Spec}\,(R[t]/(t^n - 1))$ with the natural multiplicative group structure on $\mu_n(B) = \{\, b \in B : b^n = 1 \,\}$ for each R-algebra B. In fact, we can regard μ_n as a closed subgroup scheme of $\mathbb{G}_m$ by the map $R[t, t^{-1}] \twoheadrightarrow R[t]/(t^n - 1)$ with the comultiplication, counit, and coinverse as in (2).
(4) If R has characteristic p, then we have a group scheme $\alpha_p := \mathrm{Spec}\,(R[t]/t^p)$ with the natural additive group structure on $\alpha_p(B) = \{\, b \in B : b^p = 0 \,\}$ for each R-algebra B. In fact, we can regard α_p as a closed subgroup scheme of $\mathbb{G}_a$ by the map $R[t] \twoheadrightarrow R[t]/(t^p)$ with the comultiplication, counit, and coinverse as in (1).
(5) If $\mathcal{A}$ is an abelian scheme over R, its *n-torsion subgroup* $\mathcal{A}[n] := \ker([n]_{\mathcal{A}})$ is an affine group scheme over R since $[n]_{\mathcal{A}}$ is a finite morphism.
(6) If M is an abstract group, the *constant group scheme* on M over R is a scheme $\underline{M} := \coprod_{m \in M} \mathrm{Spec}\,(R) \simeq \mathrm{Spec}\,(A)$, where $A \simeq \prod_{m \in M} R$ is the ring of R-valued functions on M, with the natural group structure (induced by M) on
$$\underline{M}(B) = \{ \text{ locally constant functions } \mathrm{Spec}\,(B) \to M \,\}$$
for each R-algebra B. Note that $A \otimes_R A$ is identified with the ring of R-values functions on $M \times M$. The comultiplication, counit, and coinverse are given by
$$\mu(f)(m, m') = f(mm'), \qquad \epsilon(f) = f(1_M), \qquad \iota(f)(m) = f(m^{-1}).$$

Let us now introduce the objects of main interest for this section. For the rest of this section, we assume that R is noetherian unless stated otherwise.

Definition 1.1.7. Let $G = \operatorname{Spec}(A)$ be an affine group scheme over R. We say that G is a (commutative) *finite flat group scheme of order* n if it satisfies the following conditions:

(i) G is locally free of rank n over R; that is, A is a locally free R-algebra of rank n.

(ii) G is commutative in the sense of the following commutative diagram

$$\begin{array}{ccc} G \times_R G & \xrightarrow{(x,y)\mapsto(y,x)} & G \times_R G \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & G & \end{array}$$

where m denotes the multiplication of G.

Remark. As a reality check, we have the following facts:

(1) G satisfies (i) if and only if the structure morphism $G \to \operatorname{Spec}(R)$ is finite flat.

(2) G satisfies (ii) if and only if $G(B)$ is commutative for each R-algebra B.

However, even if G is finite flat, $G(B)$ can be infinite for some R-algebra B such as an infinite product of R.

Example 1.1.8. Some of the group schemes that we introduced in Example 1.1.6 are finite flat group schemes, as easily seen by their affine descriptions.

(1) The n-th roots of unity μ_n is a finite flat group scheme of order n.

(2) The group scheme α_p is a finite flat group scheme of order p.

(3) If $\mathcal{A}$ is an abelian scheme of dimension g over R, its n-torsion subgroup $\mathcal{A}[n]$ is a finite flat group scheme of order n^{2g}.

(4) If M is an abelian group of order n, the constant group scheme $\underline{M}$ is a finite flat group scheme of order n.

Many basic properties of finite abelian groups extend to finite flat group schemes. Here we state two fundamental theorems without proof.

Theorem 1.1.9 (Grothendieck [**Gro60**])**.** *Let G be a finite flat R-group scheme, and let H be a closed finite flat subgroup scheme of G. Denote by m and n the orders of G and H over R, respectively. Then the quotient G/H exists as a finite flat group scheme of order m/n over R, thereby giving rise to a short exact sequence of group schemes*

$$\underline{0} \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow \underline{0}.$$

Theorem 1.1.10 (Deligne)**.** *Let G be a finite flat group scheme of order n over R. Then $[n]_G$ annihilates G; in other words, it factors through the unit section of G.*

Remark. It is unknown whether Theorem 1.1.10 holds if G is not assumed to be commutative.

We also note that finite flat group schemes behave well under base change.

Lemma 1.1.11. *Let $G = \operatorname{Spec}(A)$ be a finite flat group scheme over R. For any R-algebra B, G_B is a finite flat group scheme over B.*

Proof. Let μ, ϵ, and ι be the comultiplication, counit, and coinverse of G, respectively. It is straightforward to check that $G_B = \operatorname{Spec}(A \otimes_R B)$ is a group scheme with comultiplication, counit and coinverse given by $\mu \otimes 1, \epsilon \otimes 1$, and $\iota \otimes 1$. The finite flatness of G_B is immediate from the finite flatness of G. $\square$

1.2. Cartier duality

In this subsection, we discuss an important notion of duality for finite flat group schemes.

Definition 1.2.1. Let $G = \mathrm{Spec}\,(A)$ be a finite flat group scheme over R. We define the *Cartier dual* of G to be an R-group scheme $G^\vee$ with

$$G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \qquad \text{for each } R\text{-algebra } B,$$

where the group structure is induced by the multiplication map on $(\mathbb{G}_m)_B$.

Remark. We may identify $G^\vee = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ as a sheaf on the big fppf site.

Lemma 1.2.2. *Let G be a finite flat R-group scheme such that $[n]_G = 0$. Then we have*

$$G^\vee(B) = \mathrm{Hom}_{B\text{-}grp}(G_B, (\mu_n)_B).$$

PROOF. The assertion follows immediately by observing $\mu_n = \ker([n]_{\mathbb{G}_m})$. □

Theorem 1.2.3 (Cartier duality)**.** *Let $G = \mathrm{Spec}\,(A)$ be a finite flat group scheme of order n over R. We let μ, ϵ, and ι respectively denote the comultiplication, counit, and coinverse of A. In addition, we let $s : R \to A$ be the structure morphism, and $m_A : A \otimes_R A \to A$ be the ring multiplication map. Define $A^\vee := \mathrm{Hom}_{R\text{-}mod}(A, R)$ to be the dual R-module of A.*

(1) The dual maps $\mu^\vee$ and $\epsilon^\vee$ define an R-algebra structure on $A^\vee$.
(2) We have an identification $G^\vee \cong \mathrm{Spec}\,(A^\vee)$ with $m_A^\vee, s^\vee$, and $\iota^\vee$ as the comultiplication, counit, and coinverse.
(3) $G^\vee$ is a finite flat group scheme of order n over R.
(4) There is a canonical isomorphism $(G^\vee)^\vee \cong G$.

PROOF. The proof of (1) is straightforward and thus omitted here.

Let us now prove (2). It is not hard to verify that $G^\triangledown := \mathrm{Spec}\,(A^\vee)$ carries a structure of groups scheme with $m_A^\vee, s^\vee$, and $\iota^\vee$ as the comultiplication, counit, and coinverse. Let B be an arbitrary R-algebra. We wish to establish a canonical isomorphism

$$G^\vee(B) \cong G^\triangledown(B). \tag{1.1}$$

Let us write $\mu_B := \mu \otimes 1, \epsilon_B := \epsilon \otimes 1$, and $\iota_B : \iota \otimes 1$ for the comultiplication, counit, and coinverse of $A_B := A \otimes_R B$. We also write $s_B : s \otimes 1$ for the structure morphism $B \to A_B$. By the group scheme axioms, we have

$$(\epsilon_B \otimes \mathrm{id}) \circ \mu_B = \mathrm{id} \qquad \text{and} \qquad (\iota_B, \mathrm{id}) \circ \mu_B = s_B \circ \epsilon_B. \tag{1.2}$$

Now we use Definition 1.2.1 and the affine description of $\mathbb{G}_m$ given in Example 1.1.6 to obtain

$$\begin{aligned} G^\vee(B) &= \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \\ &\cong \left\{ f \in \mathrm{Hom}_{B\text{-alg}}(B[t, t^{-1}], A_B) : \mu_B(f(t)) = f(t) \otimes f(t), \epsilon_B(f(t)) = 1, \iota_B(f(t)) = f(t)^{-1} \right\} \end{aligned}$$

where the conditions on the last set come from compatibility with the comultiplications, counits, and coinverses on G_B and $(\mathbb{G}_m)_B$. Furthermore, an element of $\mathrm{Hom}_B(B[t, t^{-1}], A_B)$ is determined by its value at t, which must be a unit in A_B since t is a unit. We thus obtain

$$G^\vee(B) \cong \left\{ u \in A_B^\times : \mu_B(u) = u \otimes u, \epsilon_B(u) = 1, \iota_B(u) = u^{-1} \right\}.$$

Moreover, by (1.2) every element $u \in A_B^\times$ with $\mu_B(u) = u \otimes u$ must satisfy $\epsilon_B(u) = 1$ and $\iota_B(u) = u^{-1}$. Therefore we find an identification

$$G^\vee(B) \cong \left\{ u \in A_B^\times : \mu_B(u) = u \otimes u \right\}. \tag{1.3}$$

Meanwhile, by (1) we have a B-algebra structure on $A_B^\vee := A^\vee \otimes_R B$ defined by $\mu_B^\vee := \mu^\vee \otimes 1$ and $\epsilon_B^\vee := \epsilon^\vee \otimes 1$. We also have an identification

$$G^\triangledown(B) \cong \mathrm{Hom}_{R\text{-alg}}(A^\vee, B) \cong \mathrm{Hom}_{B\text{-alg}}(A^\vee \otimes_R B, B). \tag{1.4}$$

Let $m_B : B \otimes_B B \to B$ be the ring multiplication map on B. Note that $\mathrm{Hom}_{B\text{-alg}}(A^\vee \otimes_R B, B)$ is the set B-module homomorphisms $A^\vee \otimes_R B \to B$ through which $\mu_B^\vee$ and $\epsilon_B^\vee$ are compatible with m_B and id_B, respectively. Taking B-duals, we identify this set with the set of B-module homomorphisms $B \to A \otimes_R B = A_B$ through which $m_B^\vee$ and $\mathrm{id}_B^\vee$ are compatible with μ_B and ϵ_B. Moreover, the dual maps $m_B^\vee$ and $\mathrm{id}_B^\vee$ send 1 to $1 \otimes 1$ and 1, respectively. Since every B-module homomorphism $B \to A_B$ is determined by its value at 1, we have obtained an identification

$$\mathrm{Hom}_{B\text{-alg}}(A^\vee \otimes_R B, B) \cong \{ u \in A_B : \mu_B(u) = u \otimes u, \epsilon_B(u) = 1 \}.$$

Then by (1.2) we find

$$\mathrm{Hom}_{B\text{-alg}}(A^\vee \otimes_R B, B) \cong \left\{ u \in A_B^\times : \mu_B(u) = u \otimes u \right\},$$

which yields an identification

$$G^\triangledown(B) \cong \{ u \in A_B : \mu_B(u) = u \otimes u \} \tag{1.5}$$

by (1.4). We thus obtain the desired isomorphism (1.1) by (1.3) and (1.5), thereby completing the proof of (2).

Now (3) follows from (2) since $A^\vee$ is a free R-module of rank n by construction. We also deduce (4) from (2) using the canonical isomorphism $(A^\vee)^\vee \cong A$. □

We now exhibit some important examples of Cartier duality.

Lemma 1.2.4. *Given a finite flat group scheme G over R, the dual map of $[n]_G$ is $[n]_{G^\vee}$.*

PROOF. For an arbitrary R-algebra B, the dual map of $[n]_G$ sends each $f \in G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B)$ to $f \circ [n]_G = [n]_{G^\vee}(f)$. □

Proposition 1.2.5. *For every positive integer n, we have $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \simeq \mu_n$.*

PROOF. By the affine description given in Example 1.1.6, we can write $\underline{\mathbb{Z}/n\mathbb{Z}} \simeq \mathrm{Spec}\,(A)$ where $A \simeq \bigoplus_{i=0}^{n-1} Re_i$ with the comultiplication, counit, and coinverse given by

$$\mu(e_i) = \sum_{p+q=i} e_p \otimes e_q, \qquad \epsilon(e_i) = \begin{cases} 1 & \text{for i = 0} \\ 0 & \text{otherwise} \end{cases}, \qquad \iota(e_i) = e_{-i}.$$

Let $m_A : A \otimes_R A \to A$ and $s : R \to A$ respectively denote the ring multiplication map and structure morphism. Let $\{ e_i^\vee \}$ be the dual basis for $A^\vee := \mathrm{Hom}_{R\text{-mod}}(A, R)$ such that

$$e_i^\vee(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.2.3, we have an R-algebra structure on $A^\vee$, defined by $\mu^\vee$ and $\epsilon^\vee$, and a group scheme structure on $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \mathrm{Spec}\,(A^\vee)$ with $m_A^\vee, s^\vee$, and $\iota^\vee$ as the comultiplication, counit, and coinverse. In addition, it is not hard to see that the dual maps are given by

$$\mu^\vee(e_i^\vee \otimes e_j^\vee) = e_{i+j}^\vee, \quad \epsilon^\vee(1) = e_0^\vee, \quad m_A^\vee(e_i^\vee) = e_i^\vee \otimes e_i^\vee, \quad s^\vee(e_i) = 1, \quad \iota^\vee(e_i^\vee) = e_{-i}^\vee.$$

Hence, by the affine description given in Example 1.1.6, the map $A^\vee \to R[t]/(t^n - 1)$ given by $e_i^\vee \mapsto t^i$ induces an isomorphism of R-group schemes $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \simeq \mu_n$ as desired. □

Proposition 1.2.6. *Suppose that R has characteristic p. Then the R-group scheme α_p is self-dual.*

Proof. By the affine description given in Example 1.1.6, we have $\alpha_p = \mathrm{Spec}\,(R[t]/(t^p))$ with the comultiplication, counit, and coinverse given by

$$\mu(t^i) = \sum_{p+q=i} \binom{i}{p} t^p \otimes t^q, \qquad \epsilon(t^i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}, \qquad \iota(t^i) = (-t)^i.$$

Let us set $A := R[t]/(t^p)$ for notational simplicity. Let $m_A : A \otimes_R A \to A$ and $s : R \to A$ respectively denote the ring multiplication map and structure morphism. Let $\{ f_i \}$ be the dual basis for $A^\vee := \mathrm{Hom}_{R\text{-mod}}(A, R)$ such that

$$f_i(t^j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 1.2.3, we have an R-algebra structure on $A^\vee$, defined by $\mu^\vee$ and $\epsilon^\vee$, and a group scheme structure on $\alpha_p^\vee \cong \mathrm{Spec}\,(A^\vee)$ with $m_A^\vee, s^\vee$, and $\iota^\vee$ as the comultiplication, counit, and coinverse. In addition, it is not hard to see that the dual maps are given by

$$\mu^\vee(f_i \otimes f_j) = \binom{i+j}{i} f_{i+j}, \quad \epsilon^\vee(1) = 0,$$

$$m_A^\vee(f_i) = \sum_{p+q=i} f_p \otimes f_q, \quad s^\vee(f_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \iota^\vee(f_i) = (-1)^i f_i.$$

Hence the ring homomorphism $A^\vee \to A$ given by $f_i \mapsto t^i/i!$ induces an isomorphism of group schemes $\alpha_p^\vee \simeq \alpha_p$ as desired. □

Remark. When R has characteristic p, the underlying schemes of μ_p and α_p are isomorphic as we have a ring isomorphism $R[t](t^p) \to R[t]/(t^p - 1)$ given by $t \mapsto t + 1$. Propositions 1.2.5 and 1.2.6 together show that they are not isomorphic as group schemes.

Proposition 1.2.7. *Let $f : \mathcal{A} \to \mathcal{B}$ be an isogeny of abelian schemes over a ring R. Then the kernel of the dual map $f^\vee$ is naturally isomorphic to the Cartier dual of* $\ker(f)$.

Proof. By definition, we have an exact sequence

$$\underline{0} \longrightarrow \ker(f) \longrightarrow \mathcal{A} \xrightarrow{f} \mathcal{B} \longrightarrow \underline{0}$$

which gives rise to a long exact sequence

$$\underline{0} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{B}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Hom}}(\ker(f), \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{B}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m).$$

Note that the first two group schemes are trivial; in fact, abelian schemes are proper and thus admit no nontrivial maps to any affine scheme. We also have identifications

$$\underline{\mathrm{Hom}}(\ker(f), \mathbb{G}_m) \cong \ker(f)^\vee, \qquad \underline{\mathrm{Ext}}^1(\mathcal{B}, \mathbb{G}_m) \cong \mathcal{B}^\vee, \qquad \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m) \cong \mathcal{A}^\vee$$

where $\mathcal{A}^\vee$ and $\mathcal{B}^\vee$ denote the dual abelian schemes of $\mathcal{A}$ and $\mathcal{B}$, respectively. Furthermore, we may identify the last arrow in the above sequence as $f^\vee$. We thus obtain an exact sequence

$$\underline{0} \longrightarrow \ker(f)^\vee \longrightarrow \mathcal{B}^\vee \xrightarrow{f^\vee} \mathcal{A}$$

which yields the desired isomorphism $\ker(f)^\vee \cong \ker(f^\vee)$. □

Corollary 1.2.8. *Given an abelian scheme $\mathcal{A}$ over a ring R with the dual abelian scheme $\mathcal{A}^\vee$, we have a natural isomorphism $\mathcal{A}[n]^\vee \cong \mathcal{A}^\vee[n]$.*

Let us conclude this subsection by the exactness of Cartier duality.

Lemma 1.2.9. *Let $f : H \hookrightarrow G$ be a closed embedding of R-group schemes. Then we have $\ker(f^\vee) \cong (G/H)^\vee$, where $f^\vee$ denotes the dual map of f.*

PROOF. For each R-algebra B we get

$$\begin{aligned}\ker(f^\vee)(B) &= \ker\left(\mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \xrightarrow{f} \mathrm{Hom}_{B\text{-grp}}(H_B, (\mathbb{G}_m)_B)\right)\\ &\cong \mathrm{Hom}_{B\text{-grp}}((G/H)_B, (\mathbb{G}_m)_B) = (G/H)^\vee(B)\end{aligned}$$

by the universal property of the quotient group scheme $G_B/H_B \cong (G/H)_B$. □

Proposition 1.2.10. *Given a short exact sequence of finite flat R-group schemes*

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the Cartier duality gives rise to a short exact sequence

$$\underline{0} \longrightarrow G''^\vee \longrightarrow G^\vee \longrightarrow G'^\vee \longrightarrow \underline{0}.$$

PROOF. Let f and g respectively denote the maps $G' \to G$ and $G \to G''$ in the given short exact sequence, and let $f^\vee$ and $g^\vee$ denote their dual maps. Injectivity of $g^\vee$ is easy to verify using surjectivity of g and Definition 1.2.1. In addition, Lemma 1.2.9 yields $\ker(f^\vee) \cong G''^\vee$. Hence it remains to prove that $f^\vee$ is surjective. Since $G''^\vee \cong \ker(f^\vee)$ is a closed subgroup of $G^\vee$, we have a quotient $G^\vee/G''^\vee$ as a finite flat group scheme by Theorem 1.1.9. Then $f^\vee$ gives rise to a homomorphism $G^\vee/G''^\vee \to G'^\vee$. This is an isomorphism since its dual map

$$G' \longrightarrow (G^\vee/G''^\vee)^\vee \cong \ker((g^\vee)^\vee) \cong \ker(g)$$

is an isomorphism by the given exact sequence, where we use Lemma 1.2.9 for the identification $(G^\vee/G''^\vee)^\vee \cong \ker((g^\vee)^\vee)$. Hence we obtain surjectivity of $f^\vee$ as desired. □

1.3. Finite étale group schemes

In this subsection, we discuss several basic facts about finite étale group schemes. Such group schemes naturally arise in the study of Galois representations by the following fact:

Proposition 1.3.1. *Assume that R is a henselian local ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. There is an equivalence of categories*

$$\{\textit{ finite étale group schemes over } R\,\} \xrightarrow{\sim} \{\textit{ finite abelian groups with a continuous } \Gamma_k\textit{-action}\,\}$$

defined by $G \mapsto G(k^{\mathrm{sep}})$.

PROOF. Let $\overline{\mathfrak{m}} : \mathrm{Spec}\,(\overline{k}) \to \mathrm{Spec}\,(R)$ denote the geometric point associated to $\mathfrak{m} \in \mathrm{Spec}\,(R)$. Then $\Gamma_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ is identified with the étale fundamental group $\pi_1(\mathrm{Spec}\,(R), \overline{\mathfrak{m}})$. Hence we have an equivalence of categories

$$\{\text{ finite étale schemes over } R\,\} \xrightarrow{\sim} \{\text{ finite sets with a continuous } \Gamma_k\text{-action}\,\}$$

defined by $T \mapsto T(k^{\mathrm{sep}})$. The desired equivalence follows by passing to the corresponding categories of commutative group objects. □

Remark. It is not hard to see that the functor in Proposition 1.3.1 is compatible with the notion of order in both categories. Hence Proposition 1.3.1 provides an effective way to study finite étale group schemes in terms of finite groups.

Corollary 1.3.2. *If R is a henselian local ring with the residue field k, the special fiber functor yields an equivalence of categories*

$$\{\textit{ finite étale group schemes over } R\,\} \xrightarrow{\sim} \{\textit{ finite étale group schemes over } k\,\}.$$

Let us now explain a very useful criteria for étaleness of finite flat group schemes.

Definition 1.3.3. Let $G = \operatorname{Spec}(A)$ be an affine group scheme over R. We define the *augmentation ideal* of G to be the kernel of the counit $\epsilon : A \to R$.

Lemma 1.3.4. *Let $G = \operatorname{Spec}(A)$ be an affine group scheme over R with the augmentation ideal I. Then $A \simeq R \oplus I$ as an R-module.*

Proof. The assertion follows from the observation that the structure morphism $R \to A$ splits the exact sequence $0 \longrightarrow I \longrightarrow A \stackrel{\epsilon}{\longrightarrow} R \longrightarrow 0$. $\square$

Proposition 1.3.5. *Let $G = \operatorname{Spec}(A)$ be an affine group scheme over R with the augmentation ideal I. Then we have $I/I^2 \otimes_R A \simeq \Omega_{A/R}$ and $I/I^2 \simeq \Omega_{A/R} \otimes_A A/I$.*

Proof. Let us write m, e and i respectively for the multiplication, unit section and inverse of G. We have a commutative diagram

$$\begin{array}{ccc} G \times_R G & \xrightarrow{(g,h)\mapsto(g,gh^{-1})} & G \times_R G \\ & {\scriptstyle \Delta}\nwarrow \quad \nearrow{\scriptstyle (\mathrm{id},e)} & \\ & G & \end{array}$$

where the horizontal map can be also written as $(\mathrm{pr}_1, m)\circ(\mathrm{id}, i)$. We verify that the horizontal map is an isomorphism by writing down the inverse map $(x, y) \mapsto (x, y^{-1}x)$.

Let us now consider the induced commutative diagram on the level of R-algebras

$$\begin{array}{ccc} A \otimes_R A & \xleftarrow{\ \sim\ } & A \otimes_R A \\ & {\scriptstyle x\otimes y\mapsto xy}\searrow \quad \swarrow{\scriptstyle x\otimes y\mapsto x\cdot\epsilon(y)} & \\ & A & \end{array}$$

where ϵ denotes the counit of G. Let J denote the kernel of the left downward map. Then we have an identification

$$\Omega_{A/R} \cong J/J^2. \tag{1.6}$$

Moreover, as Lemma 1.3.4 yields a decomposition

$$A \otimes_R A \simeq A \otimes_R R \oplus A \otimes_R I,$$

we deduce that the kernel of the right downward map is $A \otimes_R I$. Hence the horizontal map induces an isomorphism between the two kernels $J \simeq A \otimes_R I$, which also yields an isomorphism $J^2 \simeq (A \otimes_R I)^2 \cong A \otimes_R I^2$. We thus have

$$J/J^2 \simeq (A \otimes_R I)/(A \otimes_R I^2) = A \otimes_R (I/I^2),$$

thereby obtaining a desired isomorphism $\Omega_{A/R} \simeq A \otimes_R (I/I^2)$ by (1.6). We then complete the proof by observing

$$\Omega_{A/R} \otimes_A (A/I) \simeq \big((I/I^2) \otimes_R A\big) \otimes_A A/I \cong (I/I^2) \otimes_R A/I \simeq I/I^2$$

where the last isomorphism follows from the fact that $A/I \simeq R$. $\square$

Remark. The multiplication map on G defines a natural action on $\Omega_{A/R}$. We can geometrically interpret the statement of Proposition 1.3.5 as follows:

(1) An invariant form under this action should be determined by its value along the unit section, or equivalently its image in I/I^2.
(2) An arbitrary form should be written as a function on G times an invariant form.

Corollary 1.3.6. *Let $G = \mathrm{Spec}\,(A)$ be a finite flat group scheme over R with the augment ideal I. Then G is étale if and only if $I = I^2$.*

PROOF. Since G is flat over R, it is étale if and only if $\Omega_{A/R} = 0$. Hence the assertion follows from Proposition 1.3.5. □

We discuss a number of important applications of Corollary 1.3.6.

Proposition 1.3.7. *Every finite flat constant group scheme is étale.*

PROOF. Let M be a finite group. By the affine description in Example 1.1.6, we have

$$\underline{M} \simeq \mathrm{Spec}\,\left(\bigoplus_{i \in M} Re_i\right)$$

with the counit given by the projection to Re_{1_M}. Hence the augment ideal of $\underline{M}$ is given by

$$I = \bigoplus_{i \neq 1_M} Re_i.$$

Since I has its own ring structure, we find $I = I^2$. Thus $\underline{M}$ is étale by Corollary 1.3.6. □

Proposition 1.3.8. *Assume that R is an algebraically closed field of characteristic p. Then $\underline{\mathbb{Z}/p\mathbb{Z}}$ is a unique finite étale group scheme of order p. In particular, μ_p and α_p are not étale.*

PROOF. Note that $\underline{\mathbb{Z}/p\mathbb{Z}}$ is étale by Proposition 1.3.7. For uniqueness, we use Proposition 1.3.1 together with the fact that $\mathbb{Z}/p\mathbb{Z}$ is a unique group of order p. The last statement then follows by observing that μ_p and α_p are not isomorphic to $\underline{\mathbb{Z}/p\mathbb{Z}}$ for being nonreduced. □

Proposition 1.3.9. *Let G be a finite flat group scheme over R. Then G is étale if and only if the (scheme theoretic) image of the unit section is open.*

PROOF. Let us write $G = \mathrm{Spec}\,(A)$ where A is a locally free R-algebra of finite rank. Let I denote the augment ideal of G so that the (scheme theoretic) image of the unit section is $\mathrm{Spec}\,(A/I)$. By Corollary 1.3.6, G is étale if and only if $I = I^2$. It is thus enough to show that the closed embedding $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$ is an open embedding if and only if $I = I^2$.

Suppose that $I = I^2$. By Nakayama's lemma there exists an element $f \in A$ with $f - 1 \in I$ and $fI = 0$. Note that f is idempotent; indeed, we quickly check $f^2 = f(f-1) + f = f$. Now consider the natural map $A \to A_f$. This map is surjective since we have

$$\frac{a}{f^n} = \frac{af}{f^{n+1}} = \frac{af}{f} = \frac{a}{1} \qquad \text{for any } a \in A.$$

Moreover, as $fI = 0$, the last identity shows that I is contained in the kernel. Conversely, for any element a in the kernel we have $f^n a = 0$ for some n, or equivalently $fa = 0$ as f is idempotent, and consequently see that $a = -(f-1)a + fa = -(f-1)a \in I$. We thus get a ring isomorphism $A/I \cong A_f$, thereby deducing that the closed embedding $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$ is an open embedding.

For the converse, we now suppose that $\mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$ is an open embedding. Then it is a flat morphism, implying that the ring homomorphism $A \twoheadrightarrow A/I$ is also flat. Hence we obtain a short exact sequence

$$0 \longrightarrow I \otimes_A A/I \longrightarrow A \otimes_A A/I \longrightarrow A/I \otimes_A A/I \longrightarrow 0,$$

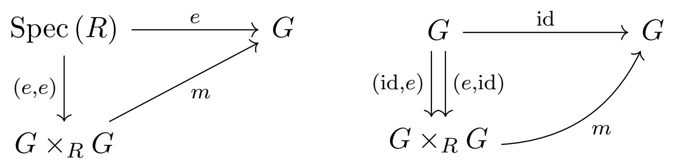
which reduces to

$$0 \longrightarrow I/I^2 \longrightarrow A/I \longrightarrow A/I \longrightarrow 0$$

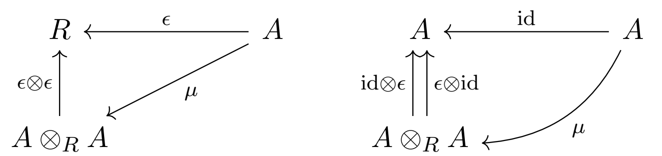
where the third arrow is the identity map. We thus deduce that $I/I^2 = 0$ as desired. □

Theorem 1.3.10. *Let G be a finite flat group scheme over R. If the order of G is invertible in R, then G is étale.*

Proof. Let us write $G = \mathrm{Spec}\,(A)$ where A is a locally free R-algebra of finite rank. As usual, we let m, e, μ, and ϵ respectively denote the multiplication map, unit section, comultiplication, and counit of G. We have commutative diagrams of schemes



which induce the following commutative diagrams of R-algebras:



(1.7)

Let $I = \ker(\epsilon)$ be the augmentation ideal of G, and let x be an arbitrary element in I. Since $\epsilon(x) = 0$, the first diagram in (1.7) implies $\mu(x) \in \ker(\epsilon \otimes \epsilon)$. Moreover, since Lemma 1.3.4 yields a decomposition

$$A \otimes_R A \simeq (R \otimes_R R) \oplus (I \otimes_R R) \oplus (R \otimes_R I) \oplus (I \otimes_R I)$$

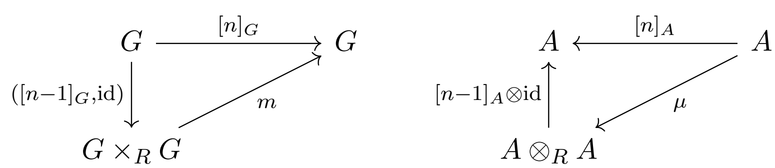
as an R-module, we deduce that

$$\ker(\epsilon \otimes \epsilon) \simeq (I \otimes_R R) \oplus (R \otimes_R I) \oplus (I \otimes_R I).$$

Hence we have $\mu(x) \in a \otimes 1 + 1 \otimes b + I \otimes_R I$ for some $a, b \in I$. Then we find $a = b = x$ using the second diagram of (1.7), thereby deducing

$$\mu(x) \in x \otimes 1 + 1 \otimes x + I \otimes_R I. \tag{1.8}$$

We assert that $[n]_G$ for each $n \geq 1$ acts as multiplication by n on I/I^2. For each $n \geq 1$, let $[n]_A : A \to A$ denote the R-algebra map induced by $[n]_G$. We have commutative diagrams



The second diagram and (1.8) together yield

$$[n]_A(x) \in [n-1]_A(x) + x + I^2 \qquad \text{for each } x \in I.$$

Since $[1]_A = \mathrm{id}_A$, the desired assertion follows by induction.

Now we let m be the order of G. Since $[m]_G$ factors through the unit section of G by Theorem 1.1.10, its induced map on $\Omega_{A/R}$ factors as $\Omega_{A/R} \to \Omega_{R/R} \to \Omega_{A/R}$. As $\Omega_{R/R} = 0$, we deduce that $[m]_G$ induces a zero map on $\Omega_{A/R}$, and also on $\Omega_{A/R} \otimes_A A/I \simeq I/I^2$ by Proposition 1.3.5. On the other hand, as noted in the preceding paragraph $[m]_G$ acts as a multiplication by m on I/I^2, which is an isomorphism if m is invertible in R. Hence we have $I/I^2 = 0$ if m is invertible in R, thereby completing the proof by Corollary 1.3.6. □

Corollary 1.3.11. *Every finite flat group scheme over a field of characteristic 0 is étale.*

1.4. The connected-étale sequence

For this subsection, we assume that R is a henselian local ring with residue field k. Under this assumption, we have a number of useful criteria for connectedness or étaleness of finite flat R-group schemes.

Lemma 1.4.1. *A finite flat R-scheme is étale if and only if its special fiber is étale.*

PROOF. This is immediate from a general fact as stated in [**Sta**, Tag 02GM]. □

Lemma 1.4.2. *Let T be a finite scheme over R. Then the following conditions are equivalent:*

(i) T is connected.
(ii) T is a spectrum of a henselian local finite R-algebra.
(iii) The action of Γ_k on $T(\overline{k})$ is transitive.

PROOF. Let us write $T \simeq \mathrm{Spec}\,(B)$ where B is a finite R-algebra. Since R is a henselian local ring, we have

$$B \simeq \prod_{i=1}^{n} B_i$$

where each B_i is a henselian local ring. Note that each $T_i := \mathrm{Spec}\,(B_i)$ corresponds to a connected component of T. Hence we see that (i) implies (ii). Conversely, (ii) implies (i) since the spectrum of a local ring is connected.

Let k_i denote the residue field of B_i for each i. Then we have

$$T(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(B, \overline{k}) \cong \coprod_{i=1}^{n} \mathrm{Hom}_k(k_i, \overline{k})$$

where Γ_k acts through $\overline{k}$. Since each $\mathrm{Hom}_k(k_i, \overline{k})$ is the orbit of the action of Γ_k, we deduce the equivalence between (i) and (iii). □

Corollary 1.4.3. *A finite R-scheme is connected if and only if its special fiber is connected.*

Remark. This is a special case of SGA 4 1/2, Exp. 1, Proposition 4.2.1, which says that for every proper R-scheme the special fiber functor induces a bijection between the connected components.

Definition 1.4.4. Given a finite flat group scheme G over R, we denote by G° the connected component of the unit section.

Proposition 1.4.5. *For a finite flat R-group scheme G, we have $G^\circ(\overline{k}) = 0$.*

PROOF. As usual, we write $G = \mathrm{Spec}\,(A)$ with some free R-algebra A of finite rank. By Lemma 1.4.2, we have $G^\circ = \mathrm{Spec}\,(A^\circ)$ for some henselian local free R-algebra A° of finite rank. As the unit section factors through G°, it induces a surjective ring homomorphism $A^\circ \to R$. Denoting its kernel by J, we obtain an isomorphism $A^\circ/J \simeq R$, which induces an isomorphism between the residue fields of A° and R. We thus find that

$$G^\circ(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(A^\circ, \overline{k}) \cong \mathrm{Hom}_k(k, \overline{k}) = 0$$

as desired. □

Theorem 1.4.6. *Let G be a finite flat group scheme over R. Then G° is a closed subgroup scheme of G such that the quotient $G^{\acute{e}t} := G/G^\circ$ is étale, thereby giving rise to a short exact sequence of finite flat group schemes*

$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\acute{e}t} \longrightarrow \underline{0}.$$

Proof. Proposition 1.4.5 implies that $(G^\circ \times_R G^\circ)(\overline{k}) \cong G^\circ(\overline{k}) \times G^\circ(\overline{k})$ is trivial. Therefore $G^\circ \times_R G^\circ$ is connected by Lemma 1.4.2.

We assert that G° is a closed subgroup of G. By construction, the unit section of G factors through G°. Moreover, as $G^\circ \times_R G^\circ$ is connected, its image under the multiplication map is a connected subscheme of G containing the unit section, and thus lies in G°. Similarly, the inverse of G maps G° into itself by connectedness. We thus obtain the desired assertion.

Since G° is a closed subgroup of G, the quotient $G^{\text{ét}} = G/G^\circ$ is a finite flat group scheme. Its unit section G°/G° has an open image as the connected component G° is open in G by the noetherian hypothesis on R. Hence we find that $G^{\text{ét}}$ is étale by Proposition 1.3.9, thereby completing the proof. □

Remark. We make several remarks about Theorem 1.4.6 and its proof.

(1) Theorem 1.4.6 essentially reduces the study of finite flat group schemes over R to two cases, namely the connected case and the étale case. We have seen in the previous subsection that finite étale group schemes are relatively easy to understand (in terms of finite groups with a Galois action). Hence most technical difficulties for us will arise in trying to understand (a system of) connected finite flat group schemes.

(2) Theorem 1.4.6 also holds when G is not commutative. To see this, we only have to prove that G° is a normal subgroup scheme of G. Let us consider the map

$$\nu : G^\circ \times_R G \to G$$

defined by $(g, h) \mapsto hgh^{-1}$. Let H be an arbitrary connected component of G. As $G^\circ \times_R H$ is connected by Lemma 1.4.2 and Proposition 1.4.5, its image under ν is a connected subscheme of G containing the unit section, and thus lies in G°. Since G is a disjoint union of its connected component, we find that the image of ν lies in G°, thereby deducing the desired assertion.

(3) We present an alternative proof of the fact that $G^\circ \times_R G^\circ$ is connected. By Corollary 1.4.3, connectedness of G° implies connectedness of G°_k. Moreover, the image of the unit section yields a k-point in G°_k. Hence G°_k is geometrically connected by a general fact as stated in [**Sta**, Tag 04KV]. Then another general fact as stated in [**Sta**, Tag 0385] implies that $G^\circ_k \times_{\mathrm{Spec}\,(k)} G^\circ_k$ is connected. We thus deduce the desired assertion by Corollary 1.4.3.

Definition 1.4.7. Given a finite flat group scheme G over R, we refer to the exact sequence in Theorem 1.4.6 as the *connected-étale sequence* of G.

Corollary 1.4.8. *A finite flat scheme G is connected if and only if $G(\overline{k}) = 0$.*

Proof. This follows from Lemma 1.4.2, Proposition 1.4.5, and Theorem 1.4.6. □

Corollary 1.4.9. *A finite flat group scheme G over R is étale if and only if $G^\circ = \underline{0}$.*

Proof. If $G^\circ = \underline{0}$, then G is étale by Theorem 1.4.6. Conversely, if G is étale the (scheme theoretic) image of the unit section is closed by definition and open by Proposition 1.3.9, thereby implying that G° is precisely the image of the unit section. □

Corollary 1.4.10. *Let $f : G \to H$ be a homomorphism of finite flat R-group schemes with H étale. Then f uniquely factors through $G^{\text{ét}} := G/G^\circ$.*

Proof. The image of G° should lie in H°, which is trivial by Corollary 1.4.9. Hence the assertion follows from the universal property of the quotient $G^{\text{ét}} = G/G^\circ$. □

Proposition 1.4.11. *Assume that $R = k$ is a perfect field. For every finite flat group k-scheme G, the connected-étale sequence canonically splits.*

PROOF. Let us write $G^{\text{ét}} := G/G^\circ$ as in Theorem 1.4.6. We wish to prove that the homomorphism $G \twoheadrightarrow G^{\text{ét}}$ admits a section. If k has characteristic 0, the assertion is obvious by Corollary 1.3.11 and Corollary 1.4.9. Hence we may assume that k has characteristic p.

As usual, we write $G = \operatorname{Spec}(A)$ with some free k-algebra A of finite rank. Let G^{red} be the reduction of G; in other words, $G^{\text{red}} = \operatorname{Spec}(A/\mathfrak{n})$ where $\mathfrak{n}$ denotes the nilradical of A. Since k is perfect, the product of two reduced k-schemes is reduced by some general facts as stated in [**Sta**, Tag 020I] and [**Sta**, Tag 035Z]. In particular, $G^{\text{red}} \times_k G^{\text{red}}$ must be reduced. Hence its image under the multiplication map should factor through G^{red}. Similarly, the inverse of G maps G^{red} into itself by reducedness. In addition, the unit section of G factors through G^{red} as k is reduced. We thus deduce that G^{red} is a closed subgroup of G.

Note that G^{red} is étale for being finite and reduced over k. Hence it suffices to prove that the homomorphism $G \twoheadrightarrow G^{\text{ét}}$ induces an isomorphism $G^{\text{red}} \simeq G^{\text{ét}}$. We have an identification $G^{\text{red}}(\overline{k})$ by reducedness of $\overline{k}$. Moreover, the homomorphism $G \twoheadrightarrow G^{\text{ét}}$ induces an isomorphism induces an isomorphism $G(\overline{k}) \simeq G^{\text{ét}}(\overline{k})$ by Theorem 1.4.6 and Corollary 1.4.8. We thus find that the homomorphism $G^{\text{red}} \simeq G^{\text{ét}}$ induces an isomorphism $G^{\text{red}}(\overline{k}) \simeq G^{\text{ét}}(\overline{k})$ which is clearly Γ_k-equivariant. The desired assertion now follows by Proposition 1.3.1. □

Remark. Interested readers can find an example of non-split connected-étale sequence over a non-perfect field in [**Pin**, §15].

Example 1.4.12. Let E be an elliptic curve over $\overline{\mathbb{F}}_p$. By Theorem 1.4.6, the group scheme $E[p]$ admits a connected-étale sequence

$$\underline{0} \longrightarrow E[p]^\circ \longrightarrow E[p] \longrightarrow E[p]^{\text{ét}} \longrightarrow \underline{0}.$$

Moreover, we have $E[p](\overline{\mathbb{F}}_p) \simeq E[p]^{\text{ét}}(\overline{\mathbb{F}}_p)$ by Proposition 1.4.5. Hence Proposition 1.3.1 implies that $E[p]^{\text{ét}}$ has order 1 when E is supersingular and order p when E is ordinary.

Let us now assume that E is ordinary. We have $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$ by Proposition 1.3.8, and thus obtain

$$\mu_p \simeq (\underline{\mathbb{Z}/p\mathbb{Z}})^\vee \hookrightarrow E[p]^\vee \simeq E^\vee[p] \simeq E[p]$$

by Proposition 1.2.5, Proposition 1.2.10, Corollary 1.2.8 and self-duality of E. Since μ_p is of order p and not étale as noted in Proposition 1.3.8, it must be connected by Theorem 1.4.6. We thus have an embedding $\mu_p \hookrightarrow E[p]^\circ$, which must be an isomorphism by order consideration. Hence the connected-étale sequence for $E[p]$ becomes

$$\underline{0} \longrightarrow \mu_p \longrightarrow E[p] \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \underline{0}.$$

We thus find $E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p\mathbb{Z}}$ by Proposition 1.4.11.

Remark. If E is supersingular, it is quite difficult to describe the p-torsion subgroup scheme $E[p]$. Note that $E[p]$ must be a self-dual connected finite flat group scheme of order p^2 over $\overline{\mathbb{F}}_p$. It is known that the only simple objects in the category of finite flat group schemes over $\overline{\mathbb{F}}_p$ are $\mu_p, \alpha_p, \underline{\mathbb{Z}/p\mathbb{Z}}$, and $\underline{\mathbb{Z}/\ell\mathbb{Z}}$ for all $\ell \neq p$. In particular, α_p is the only connected simple object with connected Cartier dual. Hence $E[p]$ should fit into an exact sequence

$$\underline{0} \longrightarrow \alpha_p \longrightarrow E[p] \longrightarrow \alpha_p \longrightarrow \underline{0}.$$

It turns out that $E[p]$ is a unique self-dual finite flat group scheme over $\overline{\mathbb{F}}_p$ which arises as a non-splitting self-extension of α_p.

1.5. The Frobenius morphism

For this subsection, we assume that $R = k$ is a field of characteristic p. We let σ denote the Frobenius endomorphism of k.

We introduce several crucial notions for studying finite flat group schemes over k.

Definition 1.5.1. Let $T = \mathrm{Spec}\,(B)$ be an affine k-scheme.

(1) We define the *Frobenius twist* of T by $T^{(p)} := T \times_{k,\sigma} k$. In other words, $T^{(p)}$ fits into the cartesian diagram

$$\begin{array}{ccc} T^{(p)} & \longrightarrow & T \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}\,(k) & \longrightarrow & \mathrm{Spec}\,(k) \end{array}$$

where the bottom map is induced by σ.

(2) The *absolute Frobenius* of T is the morphism $\mathrm{Frob}_T : T \to T$ induced by the p-th power map on B.

(3) The *relative Frobenius* of T (over k) is the morphism $\varphi_T : T \to T^{(p)} = T \times_{\mathrm{Spec}\,(k),\sigma} k$ defined by (Frob_T, s) where s denotes the structure morphism of T over k.

(4) For any $r \geq 1$, we inductively define the *p^r-Frobenius twist* and the *relative p^r-Frobenius* of T as follows:

$$T^{(p^r)} := \left(T^{(p^{r-1})}\right)^{(p)} \qquad \text{and} \qquad \varphi_T^r := \varphi_{T^{(p^{r-1})}} \circ \varphi_T^{r-1}.$$

Lemma 1.5.2. *Let $T = \mathrm{Spec}\,(B)$ be an affine k-scheme. Then φ_T^r is induced by the k-algebra homomorphism $B^{(p^r)} := B \otimes_{k,\sigma^r} k \to B$ defined by $x \otimes c \mapsto c \cdot x^{p^r}$.*

PROOF. The assertion follows from alternative identifications

$$T^{(p^r)} \cong T \times_{k,\sigma^r} k \qquad \text{and} \qquad \varphi_T^r = (\mathrm{Frob}_T^r, s) : T \to T^{(p^r)} \cong T \times_{k,\sigma^r} k$$

where s denotes the structure morphism of T over k. □

Lemma 1.5.3. *Let T and U be k-schemes.*

(a) We have identifications $(T \times_k U)^{(p)} \cong T^{(p)} \times_k U^{(p)}$ and $\varphi_{(T \times_k U)} = (\varphi_T, \varphi_U)$.

(b) Any k-scheme morphism $T \to U$ yields a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\varphi_T} & T^{(p)} \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{\varphi_U} & U^{(p)} \end{array}$$

where the second vertical arrow is the induced by the first vertical arrow.

PROOF. Considering the Frobenius twist as a functor on k-schemes, both statements are straightforward to verify using Definition 1.5.1. □

Corollary 1.5.4. *Let G be a finite flat k-scheme, and let $q = p^r$ for some $r \geq 1$.*

(1) The q-Frobenius twist $G^{(q)}$ is a finite flat k-group scheme.

(2) The relative q-Frobenius φ_G^r is a group scheme homomorphism.

PROOF. By induction, we immediately reduce to the case $p = q$. Then the desired assertions easily follow from Lemma 1.5.3 □

Definition 1.5.5. Let G be a finite flat k-group scheme. We define the *Verschiebung* of G by $\psi_G := \varphi_{G^\vee}^\vee$, i.e., the dual map of the relative Frobenius of $G^\vee$.

Remark. From the affine description of the Frobenius twist as noted in Lemma 1.5.2, we obtain a natural identification $\big((G^\vee)^{(p)}\big)^\vee \cong G^{(p)}$. We can thus regard ψ_G as a homomorphism from $G^{(p)}$ to G.

Lemma 1.5.6. *Let G and H be finite flat k-group schemes.*

(a) We have an identification $\psi_{(G\times_k H)} = (\psi_G, \psi_H)$.
(b) Any k-group scheme homomorphism $G \to H$ yields a commutative diagram

$$\begin{array}{ccc} G & \xleftarrow{\psi_G} & G^{(p)} \\ \downarrow & & \downarrow \\ H & \xleftarrow{\psi_H} & H^{(p)} \end{array}$$

where the second vertical arrow is the induced by the first vertical arrow.

Proof. This is obvious by Lemma 1.5.3 and Definition 1.5.5 □

Proposition 1.5.7. *We have the following statements:*

(1) $\varphi_{\alpha_p} = \psi_{\alpha_p} = 0$.
(2) $\varphi_{\mu_p} = 0$ and ψ_{μ_p} is an isomorphism.
(3) $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ is an isomorphism and $\psi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = 0$.

Proof. All statements are straightforward to verify using the affine descriptions from Example 1.1.6 and the duality results from Propositions 1.2.5 and 1.2.6. □

The Frobenius and Verschiebung turn out to satisfy a very simple relation.

Proposition 1.5.8. *Given a finite flat k-group scheme G, we have*

$$\psi_G \circ \varphi_G = [p]_G \qquad \text{and} \qquad \varphi_G \circ \psi_G = [p]_{G^{(p)}}.$$

Proof. The following proof is excerpted from [**Pin**, §14].

Let us write $G = \operatorname{Spec}(A)$ and $G^\vee = \operatorname{Spec}(A^\vee)$ with some free k-algebra A of finite rank. We also write $A^{(p)} := A \otimes_{k,\sigma} k$ and $(A^\vee)^{(p)} := A^\vee \otimes_{k,\sigma} k$. We let φ_A and $\varphi_{A^\vee}$ denote the k-algebra maps inducing φ_G and $\varphi_{G^\vee}$, respectively. Note that, by definition, ψ_G is induced by $\varphi_{A^\vee}^\vee$.

By the Lemma 1.5.2, the map $\varphi_A : A^{(p)} \to A$ is given by $x \otimes c \mapsto c \cdot x^p$. We also have a similar description for $\varphi_{A^\vee}$, which yields a commutative diagram

$$\begin{array}{ccccc} (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k & \xrightarrow{f\otimes c \;\mapsto\; [c\cdot f^{\otimes p}]} & \operatorname{Sym}^p A^\vee & \longrightarrow & A^\vee \\ & & \uparrow\!\uparrow & \nearrow_{\otimes f_i \;\mapsto\; \prod_{A^\vee} f_i} & \\ & & (A^\vee)^{\otimes p} & & \end{array} \tag{1.9}$$

$$\varphi_{A^\vee} : (A^\vee)^{(p)} \to A^\vee$$

where $\prod_{A^\vee}$ denotes the ring multiplication in $A^\vee$. Note that the left horizontal map is k-algebra homomorphism since k has characteristic p. By dualizing (1.9) over k, we obtain a commutative diagram

$$\begin{array}{ccccc} & & \overset{\varphi^\vee_{A^\vee}}{\frown} & & \\ A & \longrightarrow & (A^{\otimes p})^{S_p} & \xrightarrow{\lambda} & A \otimes_{k,\sigma} k = A^{(p)} \\ & \searrow & \big\downarrow & & \\ & & A^{\otimes p} & & \end{array} \tag{1.10}$$

where S_p denotes the symmetric group of order p.

Let us give an explicit description of the map λ in (1.10). It is not hard to see that any nontrivial S_p-orbit in $(A^{\otimes p})^{S_p}$ has p terms and thus maps to 0 as k has characteristic p. Hence we only need to specify $\lambda(a^{\otimes p})$ for each $a \in A$. By the isomorphism $A \cong (A^\vee)^\vee$, we may identify each $a \in A$ with $\epsilon_a \in (A^\vee)^\vee$ defined by $\epsilon_a(f) = f(a)$ for all $f \in A^\vee$. Since λ is the dual map of the left horizontal map in (1.9), for each $f \otimes c \in A^\vee \otimes_{k,\sigma} k \cong (A \otimes_{k,\sigma} k)^\vee$ we have

$$\lambda(a^{\otimes p})(f \otimes c) = (\epsilon_a)^{\otimes p}([c \cdot f^{\otimes p}]) = c \cdot f(a)^p = (f \otimes c)(a \otimes 1) = (\epsilon_a \otimes 1)(f \otimes c)$$

where the third equality follows from the identity $f(a) \otimes c = 1 \otimes c \cdot f(a)^p$ in $A \otimes_{k,\sigma} k$. We thus find $\lambda(a^{\otimes p}) = a \otimes 1$.

By our discussion in the preceding paragraph, the diagram (1.10) extends to a commutative diagram

$$\begin{array}{ccccc} & & \overset{\varphi^\vee_{A^\vee}}{\frown} & & \\ A & \longrightarrow & (A^{\otimes p})^{S_p} & \xrightarrow{\lambda} & A \otimes_{k,\sigma} k = A^{(p)} \\ & \searrow & \big\downarrow & & \big\downarrow \varphi_A \\ & & A^{\otimes p} & \xrightarrow{\otimes x_i \,\mapsto\, \prod_A x_i} & A \end{array}$$

where $\prod_A$ denotes the ring multiplication in A. Note that the diagonal map is given by the comultiplication of A, as it is the dual of the diagonal map in (1.10) given by the ring multiplication in $A^\vee$. Hence we obtain a commutative diagram of k-group schemes

$$\begin{array}{ccccc} G & & \xleftarrow{\quad\psi_G\quad} & & G^{(p)} \\ & \nwarrow^{x_1 \cdots x_p \leftarrow\!\shortmid (x_1, \cdots, x_p)} & & & \big\uparrow \varphi_G \\ & & G^{\times p} & \xleftarrow{(x, \cdots, x) \leftarrow\!\shortmid x} & G \end{array}$$

which yields $\psi_G \circ \varphi_G = [p]_G$. Then we use Lemma 1.5.3 and Lemma 1.5.6 to obtain a commutative diagram

$$\begin{array}{ccc} G^{(p)} & \xrightarrow{\varphi_{G^{(p)}}} & G^{(p^2)} \\ \psi_G \big\downarrow & & \big\downarrow \psi_{G^{(p)}} \\ G & \xrightarrow{\varphi_G} & G^{(p)} \end{array}$$

which yields $\varphi_G \circ \psi_G = \psi_{G^{(p)}} \circ \varphi_{G^{(p)}} = [p]_{G^{(p)}}$. $\square$

Let us now present a couple of important applications of the Frobenius morphism.

Proposition 1.5.9. *Let $G = \mathrm{Spec}\,(A)$ be a finite flat k-group scheme.*

(1) *G is connected if and only if φ_G^r vanishes for some r.*
(2) *G is étale if and only if φ_G is an isomorphism.*

Proof. Let I denote the augmentation ideal of A. Note that I is a maximal ideal since $A/I \simeq k$. We also have a k-space decomposition $A \simeq k \oplus I$ by Lemma 1.3.4.

Suppose that G is connected. Lemma 1.4.2 implies that A is a local ring, which is artinian for being finite over a field k. Hence its maximal ideal I is nilpotent, implying that there exists some r with $x^{p^r} = 0$ for all $x \in I$. We thus find $\varphi_G^r = 0$ by the decomposition $A \simeq k \oplus I$ and Lemma 1.5.2

Conversely, suppose that $\varphi_G^r = 0$ for some r. By observing that φ_G^r induces an isomorphism $G(\overline{k}) \simeq G^{(p)}(\overline{k})$, we find that $G(\overline{k})$ is trivial. Hence G is connected by Corollary 1.4.8. We have thus proved (1).

Next we suppose that φ_G is an isomorphism. Then φ_G induces an isomorphism on G°; in other words, φ_{G° is an isomorphism. This inductively implies that $\varphi_{(G^\circ)^{(p^r)}}$ is an isomorphism for all r, and consequently that $\varphi_{G^\circ}^r$ is an isomorphism for all r. On the other hand, we have $\varphi_{G^\circ}^r = 0$ for some r by (1). We thus find $G^\circ = 0$, which implies étaleness of G by Corollary 1.4.9.

Conversely, we assume that G is étale. Note that $\ker(\varphi_G)$ is connected by (1), which must be trivial as G° is trivial by Corollary 1.4.9. We then conclude that φ_G is an isomorphism by comparing the orders of G and $G^{(p)}$. □

Proposition 1.5.10. *The order of a connected finite flat k-group scheme is a power of p.*

Proof. Let $G = \mathrm{Spec}\,(A)$ be a connected finite flat k-group scheme of order n. We proceed by induction on n. The assertion is trivial when $n = 1$, so we only need to consider the inductive step.

Let us set $H := \ker(\varphi_G)$. Denote by I be the augmentation ideal of G, and choose elements $x_1, \cdots, x_d \in I$ which lift a basis of I/I^2. Connectedness of G implies that A is a local ring with maximal ideal I, as noted in the proof of Proposition 1.5.9. Hence $x_1, \cdots, x_d$ generate I by Nakayama's Lemma. In turn we have

$$H \simeq \mathrm{Spec}\,(A/(x_1^p, \cdots, x_d^p)) \tag{1.11}$$

by the affine description of φ_G as noted in Lemma 1.5.2. We also have $d > 0$ by Corollary 1.3.6 as G is not étale by Corollary 1.4.9.

We assert that the order of H is p^d. It suffices to show that the map

$$\lambda : k[t_1, \cdots, t_d]/(t_1^p, \cdots, t_d^p) \longrightarrow A/(x_1^p, \cdots, x_d^p)$$

defined by $t_i \mapsto x_i$ is an isomorphism. Surjectivity is clear by definition, so we only need to show injectivity. Recall that we have a k-space decomposition $A \simeq k \oplus I$ by Lemma 1.3.4. We let $\pi : A \twoheadrightarrow I/I^2$ be the natural projection map, and denote by μ the comultiplication of A. For each $j = 1, \cdots, d$, we define a k-algebra map

$$D_j : A \xrightarrow{\ \mu\ } A \otimes_k A \xrightarrow{(\mathrm{id}, \pi)} A \otimes_k I/I^2 \longrightarrow A$$

where the last arrow is induced by the map $I/I^2 \to k$ taking x_j to 1 and x_i to 0 for all $i \neq j$. Note that

$$\mu(x_i) \in x_i \otimes 1 + 1 \otimes x_i + I \otimes_k I \qquad \text{for all } i = 1, \cdots, d$$

as noted in (1.8) in the proof of Theorem 1.3.10. We thus find $\lambda \dfrac{\partial}{\partial t_j} = D_j \lambda$ as both sides agree on t_i's. This means that $\ker(\lambda)$ is stable under $\dfrac{\partial}{\partial t_j}$ for each $j = 1, \cdots, d$. In particular, every nonzero element in $\ker(\lambda)$ with minimal degree must be constant. Hence $\ker(\lambda)$ is either the zero ideal or the unit ideal. However, the latter is impossible since λ is surjective. We thus deduce that $\ker(\lambda)$ is trivial as desired.

As G is connected, we have $\varphi_G^r = 0$ for some r by Proposition 1.5.9. Then $\varphi_G^r = 0$ induces a trivial map on G/H, which means that G/H is also connected by Proposition 1.5.9. Hence its order n/p^d must be a power of p by the induction hypothesis. We thus conclude that n is a power of p as desired. □

Corollary 1.5.11. *Let G be a connected finite flat k-group scheme with the augmentation ideal I. If $\varphi_G = 0$, the order of G is p^d where d is the dimension of I/I^2 over k.*

PROOF. This follows from the proof of Proposition 1.5.10. □

Remark. Proposition 1.5.9 and Proposition 1.5.10 will be very useful for us, even when the base ring is not necessarily a field. In fact, if the base ring is a local ring with perfect residue field of characteristic p, we can check the order, connectedness, or étaleness of a finite flat group scheme by passing to the special fiber as noted in Lemma 1.4.1 and Lemma 1.4.2.

As a demonstration, we present another proof of Theorem 1.3.10 in the case where R is a local ring, without using Theorem 1.1.10. As remarked above, we may assume that R is a field by passing to the special fiber. By Corollary 1.4.9 it suffices to prove that G° is trivial. When R has characteristic p, this immediately follows from Proposition 1.5.10 by invertibility of the order. Let us now suppose that R has characteristic 0. Arguing as in the proof of Proposition 1.5.10, we can show

$$G^\circ \simeq \operatorname{Spec}(R[t_1, \cdots, t_d])$$

for some d. Then we must have $d = 0$ as G is finite over R, thereby deducing that G° is trivial as desired.

In fact, with some additional work we can even prove Theorem 1.1.10 when the base ring is a field, as explained in [**Tat97**, §3.7]. The curious reader can also find Deligne's proof of Theorem 1.1.10 in [**Sti**, §3.3]. We are also very close to a complete classification of all simple objects in the category of finite flat group schemes over $\overline{k}$ as remarked after Example 1.4.12. Instead of pursuing it here, we refer the readers to [**Sti**, Theorem 54] for a proof.

2. p-divisible groups

While finite flat group schemes have an incredibly rich theory, their structure is too simple to capture much information about p-adic Galois representations. More explicitly, as stated in Proposition 1.3.1, they are only capable of carrying information about Galois actions on finite groups. This fact leads us to consider a system of finite flat group schemes.

In this section, we develop some basic theory about p-divisible groups, which play a crucial role in many parts of p-adic Hodge theory and arithmetic geometry. While our focus is on their relation to the study of p-adic Galois representations, we also try to indicate their applications to the study of abelian varieties. The primary references for this section are Demazure's book [**Dem72**] and Tate's paper [**Tat67**].

2.1. Basic definitions and properties

Throughout this section, we let R denote a noetherian base ring.

Definition 2.1.1. Let $G = \varinjlim_{v>0} G_v$ be an inductive limit of finite flat group schemes over R with group scheme homomorphisms $i_v : G_v \to G_{v+1}$. We say that G is a *p-divisible group of height h* over R if the following conditions are satisfied:

(i) Each G_v has order p^{vh}.
(ii) Each i_v fits into an exact sequence

$$\underline{0} \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}.$$

For each v and t, we often write $G_v[p^t] := \ker([p^t]_{G_v})$.

Remark. The condition (ii) amounts to saying that each G_v is identified via i_v with $G_{v+1}[p^v]$. We may thus regard G as an fpqc sheaf where $G(T) := \varinjlim G_v(T)$ for each R-scheme T.

Example 2.1.2. We present some important examples of p-divisible groups.

(1) The *constant p-divisible group* over R is defined by $\underline{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim \underline{\mathbb{Z}/p^v\mathbb{Z}}$ with the natural inclusions. Note that the height of $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is 1.
(2) The *p-power roots of unity* over R is defined by $\mu_{p^\infty} := \varinjlim \mu_{p^v}$ with the natural inclusions. Note that the height of μ_{p^∞} is 1.
(3) Given an abelian scheme $\mathcal{A}$ over R, we define its p-divisible group by $\mathcal{A}[p^\infty] := \varinjlim \mathcal{A}[p^v]$ with the natural inclusions. The height of $\mathcal{A}[p^\infty]$ is $2g$ where g is the dimension of $\mathcal{A}$.

Remark. Another standard notation for μ_{p^∞} is $\mathbb{G}_m[p^\infty]$. Tate used a similar notation $\mathbb{G}_m(p)$ in [**Tat67**]. These notations are motivated by the identifications $\mu_{p^v} \cong \mathbb{G}_m[p^v] := \ker([p^v]_{\mathbb{G}_m})$.

Definition 2.1.3. Let $G = \varinjlim G_v$ and $H = \varinjlim H_v$ be p-divisible groups over R.

(1) A system $f = (f_v)$ of group scheme homomorphisms $f_v : G_v \to H_v$ is called a *homomorphism* from G to H if it is compatible with the transition maps for G and H in the sense of the commutative diagram

$$\begin{array}{ccc} G_v & \xrightarrow{f_v} & H_v \\ {\scriptstyle i_v}\downarrow & & \downarrow{\scriptstyle j_v} \\ G_{v+1} & \xrightarrow{f_{v+1}} & H_{v+1} \end{array}$$

where i_v and j_v are transition maps of G and H, respectively.

(2) Given a homomorphism $f = (f_v)$ from G to H, we define its *kernel* by $\ker(f) := \varinjlim \ker(f_v)$.

Example 2.1.4. Given a p-divisible group $G = \varinjlim G_v$ over R, we define the *multiplication by n* on G by a homomorphism $[n]_G := ([n]_{G_v})$.

Lemma 2.1.5. *Let $G = \varinjlim G_v$ be a p-divisible group over R. There exist homomorphisms $i_{v,t} : G_v \to G_{v+t}$ and $j_{v,t} : G_{v+t} \to G_t$ for each v and t with the following properties:*

(i) The map $i_{v,t}$ induces an isomorphism $G_v \cong G_{v+t}[p^v]$.

(ii) There exists a commutative diagram

$$\begin{array}{ccc} G_{v+t} & \xrightarrow{\quad [p^v] \quad} & G_{v+t} \\ & {\scriptstyle j_{v,t}} \searrow \quad \nearrow {\scriptstyle i_{t,v}} & \\ & G_t & \end{array}$$

(iii) We have a short exact sequence

$$\underline{0} \longrightarrow G_v \xrightarrow{i_{v,t}} G_{v+t} \xrightarrow{j_{v,t}} G_t \longrightarrow \underline{0}.$$

Proof. Let us denote the transition map $G_v \to G_{v+1}$ by i_v, and take $i_{v,t} := i_{v+t-1} \circ \cdots \circ i_v$ for each v and t. We may regard G_v as a closed subgroup scheme of G_{v+t} via $i_{v,t}$. The property (i) is then obvious for $t = 1$ by definition. For $t > 1$, we inductively proceed by observing

$$G_{v+t}[p^v] \cong G_{v+t}[p^{v+t-1}] \cap G_{v+t}[p^v] \cong G_{v+t-1} \cap G_{v+t}[p^v] \cong G_{v+t-1}[p^v].$$

Now (i) implies that each G_v is annihilated by $[p^v]$. More generally, the image of $[p^v]_{G_{v+t}}$ is annihilated by $[p^t]$ for each v and t. Hence the map $[p^v]_{G_{v+t}}$ uniquely factors over a map $j_{v,t} : G_{v+t} \to G_t$, thereby yielding a commutative diagram as stated in (ii).

We now have left exactness of the sequence in (iii) by (i) and (ii). Moreover, j_{v+t} induces a closed embedding $G_{v+t}/G_v \hookrightarrow G_t$, which is easily seen to be an isomorphism by comparing the orders. We thus deduce the exactness of the sequence in (iii). $\square$

Corollary 2.1.6. *Let $G = \varinjlim G_v$ be a p-divisible group over R.*

(1) We have an identification $G_v \cong \ker([p^v]_G)$ for each v.

(2) The homomorphism $[p]$ is surjective as a map of fpqc shaves.

Remark. Corollary 2.1.6 shows that the kernel of a homomorphism between two p-divisible groups may not be a p-divisible group.

We note some fundamental properties of p-divisible groups inherited from finite flat group schemes.

Proposition 2.1.7. *Let $G = \varinjlim G_v$ be a p-divisible group of height h over R.*

(1) For each v, we have an exact sequence

$$G_{v+1} \xrightarrow{[p^v]} G_{v+1} \xrightarrow{j_v} G_v \longrightarrow \underline{0}.$$

(2) The inductive system $G^\vee := \varinjlim G_v^\vee$ with the $j_v^\vee$ as transition maps is a p-divisible group of height h over R.

(3) There is a canonical isomorphism $(G^\vee)^\vee \cong G$.

Proof. Let us take $i_{v,t}$ and $j_{v,t}$ as in Lemma 2.1.5. Then we have a commutative diagram

$$\begin{array}{ccccccccccc} & & & & & G_1 & & & & & \\ & & & & \overset{j_{v,1}}{\nearrow} & & \overset{i_{1,v}}{\searrow} & & & & \\ \underline{0} & \longrightarrow & G_v & \xrightarrow{i_v = i_{v,1}} & G_{v+1} & \xrightarrow{\quad [p^v] \quad} & G_{v+1} & \xrightarrow{j_v = j_{1,v}} & G_v & \longrightarrow & \underline{0} \end{array}$$

where the horizontal arrows form an exact sequence. In particular, we obtain an exact sequence as stated in (1). Moreover, as $[p^v]^\vee_{G_{v+1}} = [p^v]_{G^\vee_{v+1}}$ by Lemma 1.2.4, we have a dual exact sequence

$$\underline{0} \longrightarrow G_v^\vee \xrightarrow{j_v^\vee} G_{v+1}^\vee \xrightarrow{[p^v]} G_{v+1}^\vee$$

by Proposition 1.2.10. Hence we deduce (2) and (3) by Theorem 1.2.3. □

Definition 2.1.8. Given a p-divisible group G over R, we refer to the p-divisible group $G^\vee$ in Proposition 2.1.7 as the *Cartier dual* of G.

Example 2.1.9. The Cartier duals for p-divisible groups from Example 2.1.2 are as follows:

1. (1) We have an identification $(\underline{\mathbb{Q}_p/\mathbb{Z}_p})^\vee \cong \mu_{p^\infty}$ by Proposition 1.2.5.
2. (2) Given an abelian scheme $\mathcal{A}$ over R, we have $\mathcal{A}[p^\infty]^\vee \cong \mathcal{A}^\vee[p^\infty]$ by Corollary 1.2.8 where $\mathcal{A}^\vee$ denotes the dual abelian scheme of $\mathcal{A}$.

Proposition 2.1.10. *Assume that R is a henselian local ring with residue field k. Let $G = \varinjlim G_v$ be a p-divisible group over R, and write $G_v^{\text{ét}} := G_v/G_v^\circ$ for each v. Then we have a short exact sequence of p-divisible groups*

$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow \underline{0}$$

where $G^\circ = \varinjlim G_v^\circ$ and $G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$.

Proof. Let $i_v : G_v \to G_{v+1}$ denote the transition map. It suffices to construct homomorphisms $i_v^\circ : G_v^\circ \to G_{v+1}^\circ$ and $i_v^{\text{ét}} : G_v^{\text{ét}} \to G_{v+1}^{\text{ét}}$ so that the diagram

$$\begin{array}{ccccccccc} & & \underline{0} & & \underline{0} & & \underline{0} & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \underline{0} & \longrightarrow & G_v^\circ & \longrightarrow & G_v & \longrightarrow & G_v^{\text{ét}} & \longrightarrow & \underline{0} \\ & & \downarrow{\scriptstyle i_v^\circ} & & \downarrow{\scriptstyle i_v} & & \downarrow{\scriptstyle i_v^{\text{ét}}} & & \\ \underline{0} & \longrightarrow & G_{v+1}^\circ & \longrightarrow & G_{v+1} & \longrightarrow & G_{v+1}^{\text{ét}} & \longrightarrow & \underline{0} \\ & & \downarrow{\scriptstyle [p^v]} & & \downarrow{\scriptstyle [p^v]} & & \downarrow{\scriptstyle [p^v]} & & \\ \underline{0} & \longrightarrow & G_{v+1}^\circ & \longrightarrow & G_{v+1} & \longrightarrow & G_{v+1}^{\text{ét}} & \longrightarrow & \underline{0} \end{array}$$

is commutative with exact rows and columns. Exactness of three rows directly follows from Theorem 1.4.6, while exactness of the middle column is immediate by definition. In addition, the bottom two squares clearly commute.

By Corollary 1.4.10, there is a unique choice of $i_v^{\text{ét}}$ such that the top right square commutes. We assert that the third column is exact with this choice. By Proposition 1.3.1, we may work on the level of $\overline{k}$-points. Since the first column vanishes on $\overline{k}$-points by Proposition 1.4.5, all horizontal arrows between the second and the third column become isomorphism on $\overline{k}$-points. Hence the desired exactness follows from exactness of the middle column.

Let us now regard G_v° as a subgroup of G_{v+1} via the embedding i_v. Then G_v° must lie in G_{v+1}° for being connected. Hence there exists a unique closed embedding i_v° which makes the top left square commutative.

It remains to show that the first column is exact with our choice of i_v°. As i_v° is a closed embedding by construction, we only need to show that $G_v^\circ \cong G_{v+1}^\circ[p^v]$ via i_v°. Indeed, as G_v° is a subgroup of both $G_v \cong G_{v+1}[p^v]$ and G_{v+1}°, it must be a subgroup of $G_{v+1}^\circ[p^v]$. Hence it remains to show that $G_{v+1}^\circ[p^v]$ is a subgroup of G_v°. As $G_{v+1}^\circ[p^v]$ is a subgroup of $G_{v+1}[p^v] \cong G_v$, it suffices to show that $G_{v+1}^\circ[p^v]$ is connected. Since $G_{v+1}^\circ(\overline{k}) = 0$ by Corollary 1.4.8, we have $G_{v+1}^\circ[p^v](\overline{k}) = 0$ as well. Hence $G_{v+1}^\circ[p^v]$ is connected by Corollary 1.4.8. ☐

Definition 2.1.11. Assume that $R = k$ is a field of characteristic p. Let $G = \varinjlim G_v$ be a p-divisible group over k.

(1) The *Frobenius twist* of G is an inductive limit $G^{(p)} := \varinjlim G_v^{(p)}$ where the transition maps are induced by the transition maps for G.
(2) We define the *Frobenius* of G by $\varphi_G := (\varphi_{G_v})$ and the *Verschiebung* of G by $\psi_G := (\psi_{G_v})$.

Proposition 2.1.12. *Assume that $R = k$ is a field of characteristic p. Let G be a p-divisible group of height h over k.*

(1) The Frobenius twist $G^{(p)}$ is a p-divisible group of height h over k.
(2) The Frobenius φ_G and the Verschiebung ψ_G are homomorphisms.
(3) We have $\psi_G \circ \varphi_G = [p]_G$ and $\varphi_G \circ \psi_G = [p]_{G^{(p)}}$.

PROOF. The statements (1) and (2) are straightforward to check using Lemma 1.5.3 and Lemma 1.5.6. The statement (3) is a direct consequence of Proposition 1.5.8. ☐

We finish this subsection by describing a connection between p-divisible groups and continuous Galois representations.

Definition 2.1.13. Assume that $R = k$ is a field. Given a p-divisible group $G = \varinjlim G_v$ over k, we define the *Tate module* of G by

$$T_p(G) := \varprojlim G_v(\overline{k})$$

where the transition maps are induced by the homomorphisms $j_{v,1} : G_{v+1} \twoheadrightarrow G_v$ from Lemma 2.1.5.

Proposition 2.1.14. *Assume that $R = k$ is a field with characteristic not equal to p. Then we have an equivalence of categories*

$$\{\, p\text{-divisible groups over } k \,\} \xrightarrow{\sim} \{\, \text{finite free } \mathbb{Z}_p\text{-modules with a continuous } \Gamma_k\text{-action} \,\}$$

defined by $G \mapsto T_p(G)$.

PROOF. Let us first verify that the functor is well-defined. Let $G = \varinjlim G_v$ be an arbitrary p-divisible group over k. Since G_v is killed by $[p^v]$ as noted in Lemma 2.1.5, each $G_v(\overline{k})$ is a finite free module over $\mathbb{Z}/p^v\mathbb{Z}$ with a continuous Γ_k-action. Hence $T_p = \varprojlim G_v(\overline{k})$ is a finite free $\mathbb{Z}_p$-module with a continuous Γ_k-action.

As all finite flat k-group schemes of p-power order are étale by Theorem 1.3.10, we deduce full faithfulness of the functor from Proposition 1.3.1. Hence it remains to prove essential surjectivity of the functor. Let M be a finite free $\mathbb{Z}_p$-module with a continuous Γ_k-action. As each $M_v := M/(p^v)$ gives rise to a finite étale group scheme G_v by Proposition 1.3.1, we form a p-divisible group $G = \varinjlim G_v$ with $T_p(G) = M$. ☐

2.2. Serre-Tate equivalence for connected p-divisible groups

In this subsection, we assume that R is a complete noetherian local ring with residue field k of characteristic p.

Definition 2.2.1. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) We say that G is *connected* if each G_v is connected, and *étale* if each G_v is étale.
(2) The p-divisible groups G° and $G^{\text{ét}}$ as constructed in Proposition 2.1.10 are respectively called the *connected part* and the *étale part* of G.

Example 2.2.2. Below are essential examples of étale/connected p-divisible groups.

(1) The constant p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is étale by Proposition 1.3.7.
(2) The p-power roots of unity μ_{p^∞} is connected by Corollary 1.4.8.

For the rest of this subsection, we let $\mathscr{A} := R[[t_1, \cdots, t_d]]$ denote the ring of power series over R with d variables. Note that $\mathscr{A}\widehat{\otimes}_R\mathscr{A} \cong R[[t_1, \cdots, t_d, u_1, \cdots, u_d]]$. We often write $T := (t_1, \cdots, t_d)$ and $U := (u_1, \cdots, u_d)$.

Let us introduce the key objects for studying connected p-divisible groups over R.

Definition 2.2.3. A continuous ring homomorphism $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ is called a *(commutative) formal group law of dimension* d over R if the power series $\Phi_i(T, U) := \mu(t_i) \in \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ form a family $\Phi(T, U) := (\Phi_i(T, U))$ that satisfies the axioms

(i) associativity: $\Phi(T, \Phi(U, V)) = \Phi(\Phi(T, U), V)$,
(ii) unit section: $\Phi(T, 0_d) = T = \Phi(0_d, T)$,
(iii) commutativity: $\Phi(T, U) = \Phi(U, T)$

where $V = (v_1, \cdots, v_d)$ is a tuple of d independent variables.

Example 2.2.4. The *multiplicative formal group law* over R is a 1-dimensional formal group law $\mu_{\widehat{\mathbb{G}}_m} : R[[t]] \to R[[t, u]]$ defined by $\mu_{\widehat{\mathbb{G}}_m}(t) = t + u + tu = (1 + t)(1 + u) - 1$.

Lemma 2.2.5. *Let μ be a formal group law of dimension d over R.*

(1) We have commutative diagrams

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\mu} & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu\widehat{\otimes}\mathrm{id}} \\ \mathscr{A}\widehat{\otimes}_R\mathscr{A} & \xrightarrow{\mathrm{id}\widehat{\otimes}\mu} & \mathscr{A}\widehat{\otimes}_R\mathscr{A}\widehat{\otimes}_R\mathscr{A} \end{array} \qquad \begin{array}{ccc} \mathscr{A}\widehat{\otimes}_R\mathscr{A} & \xrightarrow{x\widehat{\otimes}y\mapsto y\widehat{\otimes}x} & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle\mu}\nwarrow & & \nearrow{\scriptstyle\mu} \\ & \mathscr{A} & \end{array}$$

(2) The ring homomorphism $\epsilon : \mathscr{A} \to R$ given by $\epsilon(t_i) = 0$ fits into commutative diagrams

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\mathrm{id}} & \mathscr{A} & \xrightarrow{\sim} & \mathscr{A}\widehat{\otimes}_R R \\ & {\scriptstyle\mu}\searrow & & \nearrow{\scriptstyle\mathrm{id}\widehat{\otimes}\epsilon} & \\ & & \mathscr{A}\widehat{\otimes}_R\mathscr{A} & & \end{array} \qquad \begin{array}{ccccc} \mathscr{A} & \xrightarrow{\mathrm{id}} & \mathscr{A} & \xrightarrow{\sim} & R\widehat{\otimes}_R\mathscr{A} \\ & {\scriptstyle\mu}\searrow & & \nearrow{\scriptstyle\epsilon\widehat{\otimes}\mathrm{id}} & \\ & & \mathscr{A}\widehat{\otimes}_R\mathscr{A} & & \end{array}$$

(3) There exists a ring homomorphism $\iota : \mathscr{A} \to \mathscr{A}$ that fits into the commutative diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\mu} & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle\epsilon}\downarrow & & {\scriptstyle\mathrm{id}\widehat{\otimes}\iota}\downarrow\downarrow{\scriptstyle\iota\widehat{\otimes}\mathrm{id}} \\ R & \longrightarrow & \mathscr{A} \end{array}$$

PROOF. Let $\Phi(T, U)$ be as in Definition 2.2.3. The statements (1) and (2) immediately follow from the axioms in Definition 2.2.3. Moreover, by setting $I_i(t) := \iota(t_i)$, the statement (3) amounts to existence of a family $I(T) = (I_i(T))$ of d power series with

$$\Phi(T, I(T)) = 0 = \Phi(I(T), T).$$

By the axiom (iii) in Definition 2.2.3, we only need to consider the equation $\Phi(T, I(T)) = 0$. We wish to present the desired family as a limit $I(T) = \lim_{j\to\infty} P_j(T)$ where each P_j is a family of degree j polynomials in $t_1, \cdots, t_d$ with

(a) $P_j = P_{j-1} \bmod (t_1, \cdots, t_d)^j$,
(b) $\Phi(P_j(T), T) = 0 \bmod (t_1, \cdots, t_d)^{j+1}$.

As we have $\Phi(T, U) = T + U \bmod (t_1, \cdots, t_d, u_1, \cdots, u_d)^2$ by the axiom (ii) in Definition 2.2.3, we must set $P_1(T) := -T$. Now we inductively construct $P_j(T)$ for all $j \geq 1$. By the property (b) for P_j, there exists a unique homogeneous polynomial $\Delta_j(T)$ of degree $j+1$ with

$$\Delta_j(T) = -\Phi(P_j(T), T) \bmod (t_1, \cdots, t_d)^{j+2}.$$

Setting $P_{j+1}(T) := P_j(T) + \Delta_j(T)$, we immediately verify the property (a) for P_{j+1}, and also verify the property (b) for P_{j+1} by

$$\Phi(P_{j+1}(T), T) = \Phi(P_j(T) + \Delta_j(T), T) = \Phi(P_j(T), T) + \Delta_j(T) = 0 \bmod (t_1, \cdots, t_d)^{j+2}$$

where the second equality comes from observing $\Delta_j(T)^2 = 0 \bmod (t_1, \cdots, t_d)^{j+2}$ by degree consideration. □

Remark. Lemma 2.2.5 shows that a formal group law μ over R amounts to a formal group structure on the formal scheme $\mathrm{Spf}(\mathscr{A})$ with μ, ϵ, and ι as the comultiplication, counit, and coinverse.

Definition 2.2.6. Let μ and ν be formal group laws of dimension d over R. A continuous ring homomorphism $\gamma : \mathscr{A} \to \mathscr{A}$ is called a *homomorphism* from μ to ν if the following diagram commutes:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \nu\ } & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \gamma\widehat{\otimes}\gamma} \\ \mathscr{A} & \xrightarrow{\ \mu\ } & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \end{array}$$

Remark. Note that γ goes from the power series ring for ν to the power series ring for μ. This is so that γ corresponds to a formal group homomorphism between the formal groups associated to μ and ν in the sense of the remark after Lemma 2.2.5.

Lemma 2.2.7. *Let μ and ν be formal group laws of dimension d over R, represented by families of power series $\Phi(T, U) := (\Phi_i(T, U))$ and $\Psi(T, U) := (\Psi_i(T, U))$ with $\Phi_i(T, U) := \mu(t_i)$ and $\Psi_i(T, U) := \nu(t_i)$. A continuous ring homomorphism $\gamma : \mathscr{A} \to \mathscr{A}$ is a homomorphism from μ to ν if and only if the family $\Xi(T) = (\Xi_i(T))$ of d power series (in d variables) defined by $\Xi_i(T) := \gamma(t_i)$ satisfies $\Psi(\Xi(T), \Xi(U)) = \Xi(\Phi(T, U))$.*

PROOF. The diagram in Definition 2.2.6 becomes commutative if and only if we have $f(\Psi(\Xi(T), \Xi(U))) = f(\Xi(\Phi(T, U)))$ for every $f(T) \in \mathscr{A}$. □

Example 2.2.8. Let μ be a formal group law of dimension d over R. The *multiplication by* n on μ, denoted by $[n]_\mu$, is inductively defined by $[1]_\mu := \mathrm{id}_{\mathscr{A}}$ and $[n]_\mu := ([n-1]_\mu \widehat{\otimes} \mathrm{id}) \circ \mu$.

Remark. As expected, $[n]_\mu$ corresponds to the multiplication by n map on the formal group associated to μ in the sense of the remark after Lemma 2.2.5.

Definition 2.2.9. Let μ be a formal group law of dimension d over R.

(1) The ideal $\mathscr{I} := (t_1, \cdots, t_d)$ is called the *augmentation ideal* of μ.
(2) We say that μ is *p-divisible* if $[p]_\mu : \mathscr{A} \to \mathscr{A}$ is finite flat in the sense that it makes $\mathscr{A}$ a free module of finite rank over itself.

Remark. The ideal $\mathscr{I}$ is the kernel of the ring homomorphism $\epsilon : \mathscr{A} \to R$ from Lemma 2.2.5, which corresponds to the counit of the formal group associated to μ as remarked after Lemma 2.2.5. Hence the notion of augmentation ideal for formal group laws is coherent with the notion of augmentation ideal for affine group schemes as defined in Definition 1.3.3.

It turns out that every p-divisible formal group law yields a connected p-divisible group.

Proposition 2.2.10. *Let μ be a p-divisible formal group law over R with the augmentation ideal $\mathscr{I}$. Define $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ and $\mu[p^v] := \mathrm{Spec}\,(A_v)$ for each v.*

(1) Each $\mu[p^v]$ carries the natural structure of a connected finite flat R-group scheme.
(2) The inductive limit $\mu[p^\infty] := \varinjlim \mu[p^v]$ with the natural transition maps is a connected p-divisible group over R.

Proof. Let us take ϵ and ι as in Lemma 2.2.5. For each v, we have

$$A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I}) \cong \mathscr{A}/\mathscr{I} \otimes_{\mathscr{A},[p^v]_\mu} \mathscr{A} \cong R \otimes_{\mathscr{A},[p^v]_\mu} \mathscr{A}. \tag{2.1}$$

Hence $\mu[p^v] = \mathrm{Spec}\,(A_v)$ has the structure of an R-group scheme with $1 \otimes \mu, 1 \otimes \epsilon$, and $1 \otimes \iota$ as the comultiplication, counit, and coinverse.

Denote by r the rank of $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$ as a free module. A simple induction shows that the rank of $\mathscr{A}$ over $[p^v]_\mu(\mathscr{A})$ is r^v. We then deduce from (2.1) that A_v is finite free over R of rank r^v. Thus $\mu[p^v]$ is indeed finite flat of order r^v over R.

Moreover, as R is a local ring, the power series ring $\mathscr{A}$ is also a local ring. Hence $A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I})$ is a local ring as well. We thus deduce that $\mu[p^v]$ is connected.

By Proposition 1.5.10, the order of $\mu[p]$ is p^h for some h. Then our discussion above shows that $\mu[p^v]$ has order p^{vh}. Furthermore, the ring homomorphism

$$A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I}) \longrightarrow [p]_\mu(\mathscr{A})/[p^{v+1}]_\mu(\mathscr{I})$$

induced by $[p]$ is an isomorphism for being a surjective map between two free R-algebras of the same rank. Hence we get a surjective ring homomorphism

$$A_{v+1} = \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I}) \twoheadrightarrow [p]_\mu(\mathscr{A})/[p^{v+1}]_\mu(\mathscr{I}) \simeq A_v,$$

which induces an embedding $i_v : \mu[p^v] \hookrightarrow \mu[p^{v+1}]$. It is then straightforward to check that i_v identifies $\mu[p^v]$ as the kernel of $[p^v]$ on $\mu[p^{v+1}]$. We thus conclude that $\mu[p^\infty] := \varinjlim \mu[p^v]$ is a connected p-divisible group of height h over R. □

Remark. Let $\mathscr{G}_\mu$ denote the formal group associated to μ. Then by construction we have $\mu[p^v] \cong \mathscr{G}_\mu[p^v]$ for each v. With this observation the proof of (2) becomes almost trivial.

Definition 2.2.11. Given a p-divisible formal group law μ over R, we define its *associated connected p-divisible group* over R to be $\mu[p^\infty]$ as constructed in Proposition 2.2.10.

Example 2.2.12. Consider the multiplicative formal group law $\mu_{\widehat{\mathbb{G}}_m}$ introduced in Example 2.2.4. An easy induction shows $[p^v]_{\mu_{\widehat{\mathbb{G}}_m}}(t) = (1+t)^{p^v} - 1$ for each v. We then find $\mu_{\widehat{\mathbb{G}}_m}[p^v] \cong \mu_{p^v}$ for each v by the affine description in Example 1.1.6, thereby deducing $\mu_{\widehat{\mathbb{G}}_m}[p^\infty] = \mu_{p^\infty}$.

The association described in Proposition 2.2.10 defines a functor from the category of p-divisible formal group laws to the category of connected p-divisible groups. Our next goal is to prove a theorem of Serre and Tate that this functor is an equivalence of categories.

Proposition 2.2.13. *Let $G = \varinjlim G_v$ be a connected p-divisible group over R with $G_v = \operatorname{Spec}(A_v)$ for each v. We have a continuous isomorphism*

$$\varprojlim(A_v \otimes_R k) \simeq k[[t_1, \cdots, t_d]]$$

for some positive integer d.

Proof. Let us write $\overline{G} := G \times_R k$ and $\overline{G_v} := G_v \times_R k$. As G is connected, each $\overline{G}_v$ is connected by Corollary 1.4.3. Hence each $A_v \otimes_R k$ is a local ring by Lemma 1.4.2.

Let us take $H_v := \ker(\varphi^v_{\overline{G}})$. Note that each H_v must be a closed subgroup scheme of $\overline{G}[p^v] = \overline{G}_v$ since $\psi^v_{\overline{G}} \circ \varphi^v_{\overline{G}} = [p^v]_{\overline{G}}$ by Proposition 2.1.12. Hence we have $H_v \simeq \operatorname{Spec}(B_v)$ for some free k-algebra B_v with a surjective k-algebra map $A_v \otimes_R k \twoheadrightarrow B_v$. In addition, B_v is a local ring for being a quotient of a local ring A_v. We also note that each $\overline{G}_v$ is a subgroup of $\ker(\varphi^w_{\overline{G}}) = H_w$ for some large w by Proposition 1.5.9. In other words, for each v we have a surjective k-algebra map $B_w \twoheadrightarrow A_v \otimes_R k$. Hence we obtain a continuous isomorphism

$$\varprojlim A_v \otimes_R k \simeq \varprojlim B_v. \tag{2.2}$$

Let J_v be the augmentation ideal of H_v, and take $J := \varprojlim J_v$. By definition, we have $B_v/J_v \simeq k$. Let $y_1, \cdots, y_d$ be elements of J which lift a basis for J_1/J_1^2. As $H_1 \cong \ker(\varphi_{H_v})$ by construction, we use Lemma 1.5.2 to obtain a cartesian diagram

$$\begin{array}{ccc} k \cong (B_v/J_v) \otimes_{R,\sigma} k & \longrightarrow & B_1 \\ \uparrow & & \uparrow \\ B_v^{(p)} \cong B_v \otimes_{R,\sigma} k & \xrightarrow{x \otimes c \mapsto c \cdot x^p} & B_v \end{array}$$

which yields $B_1 \cong B_v/J_v^{(p)}$ where $J_v^{(p)}$ denotes the ideal generated by p-th powers of elements in J_v. We thus find $J_1 \cong J_v/J_v^{(p)}$ and consequently $J_1/J_1^2 \cong J_v/J_v^2$. Therefore the images of $y_1, \cdots, y_d$ form a basis of J_v/J_v^2, and thus generate the ideal J_v by Nakayama's lemma. In particular, we have a surjective k-algebra map

$$k[t_1, \cdots, t_d] \twoheadrightarrow B_v$$

which sends each t_i to the image of y_i in B_v. Furthermore, as $\varphi^v_{H_v}$ vanishes on H_v by construction, the above map induces a surjective k-algebra map

$$k[t_1, \cdots, t_d]/(t_1^{p^v}, \cdots, t_d^{p^v}) \twoheadrightarrow B_v \tag{2.3}$$

by Lemma 1.5.2. By passing to the limit we obtain a continuous ring homomorphism

$$k[[t_1, \cdots, t_d]] \twoheadrightarrow \varprojlim B_v.$$

By (2.2), it remains to prove that this map is an isomorphism. It suffices to prove that the k-algebra homomorphism (2.3) is an isomorphism for each v. By surjectivity, we only need to show that the source and the target have equal dimensions over k. In other words, it is enough to show that the dimension of B_v over k is p^{dv}, or equivalently that H_v has order p^{dv}. When $v = 1$, this is an immediate consequence of Corollary 1.5.11. Let us now proceed by induction on v. As $\varphi^{v+1}_{H_{v+1}} : H_{v+1} \to H^{(p)}_{v+1}$ vanishes, it should factor through $\ker(\varphi^v_{\overline{G}^{(p)}}) \cong H_v^{(p)}$. Moreover, as $\varphi_{\overline{G}} \circ \psi_{\overline{G}} = [p]_{\overline{G}^{(p)}}$ is surjective by Corollary 2.1.6, $\varphi_{\overline{G}}$ is also surjective. Since the preimage of $H_v^{(p)} \cong \ker(\varphi^v_{\overline{G}^{(p)}})$ under $\varphi_{\overline{G}}$ must lie in $\ker(\varphi^{v+1}_{\overline{G}}) = H_{v+1}$, we deduce that the map $H_{v+1} \to H_v^{(p)}$ induced by $\varphi_{H_{v+1}}$ is surjective. We thus obtain a short exact sequence

$$\underline{0} \longrightarrow H_1 \longrightarrow H_{v+1} \xrightarrow{\varphi_{H_{v+1}}} H_v^{(p)} \longrightarrow \underline{0}.$$

As the order of $H_v^{(p)}$ is the same as the order of H_v, we complete the induction step by the multiplicativity of orders in short exact sequences. □

Lemma 2.2.14. *Let μ be a formal group law of dimension d over R with the augmentation ideal $\mathscr{I}$. For each positive integer n, we have*

$$[n]_\mu(t_i) \in nt_i + \mathscr{I}^2.$$

PROOF. For each n, we define the family $\Xi_n(T) = (\Xi_{n,i}(T))$ of d power series in d variables by $\Xi_{n,i}(T) := [n]_\mu(t_i)$. We can rewrite the desired assertion as

$$\Xi_n(T) = nT \bmod \mathscr{I}^2. \tag{2.4}$$

Let us define the family $\Phi(T,U) = (\Phi_i(T,U))$ of d power series in $2d$ variables by $\Phi_i(T,U) := \mu(t_i)$. By the axiom (ii) in Definition 2.2.3 we have

$$\Phi(T,U) = T + U \bmod (t_1, \cdots, t_d, u_1, \cdots, u_d)^2.$$

Hence the inductive formula $[n]_\mu = ([n-1]_\mu \widehat{\otimes} \mathrm{id}) \circ \mu$ yields

$$\Xi_n(T) = \Phi(\Xi_{n-1}(T), T) = \Xi_{n-1}(T) + T \bmod \mathscr{I}^2.$$

Moreover, we have $\Xi_1(T) = T$ since $[1]_\mu = \mathrm{id}_{\mathscr{A}}$. We thus obtain (2.4) by induction on n. □

Lemma 2.2.15. *Let μ be a formal group law over R with the augmentation ideal $\mathscr{I}$. Define $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ for each v. Then we have a natural continuous isomorphism*

$$\mathscr{A} \cong \varprojlim A_v.$$

PROOF. Let us write $\mathfrak{m}$ for the maximal idea of R and $\mathfrak{M} := \mathfrak{m}\mathscr{A} + \mathscr{I}$ for the maximal ideal of $\mathscr{A}$. For each v we define $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$, which is a free local R-algebra of finite rank by Proposition 2.2.10. For each i and v we have $\mathfrak{M}^w \subseteq [p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}$ for some w since the algebra $\mathscr{A}/([p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}) = A_v/\mathfrak{m}^i A_v$ is local artinian. Moreover, by Lemma 2.2.14 we find $[p]_\mu(\mathscr{I}) \subseteq p\mathscr{I} + \mathscr{I}^2 \subseteq \mathfrak{M}\mathscr{I}$ and thus $[p]_\mu^v(\mathscr{I}) \subseteq \mathfrak{M}^v\mathscr{I}$ for all v. Hence for each i and v we have $[p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A} \subseteq \mathfrak{M}^w$ for some w. We thus obtain

$$\mathscr{A} \cong \varprojlim_w \mathscr{A}/\mathfrak{M}^w \cong \varprojlim_{i,v} \mathscr{A}/([p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}) \cong \varprojlim_{v,i} A_v/\mathfrak{m}^i A_v \cong \varprojlim_v A_v$$

where the last isomorphism comes from the fact that each A_v is $\mathfrak{m}$-adically complete for being finite free over R by a general fact as stated in [**Sta**, Tag 031B]. □

Theorem 2.2.16 (Serre-Tate)**.** *There exists an equivalence of categories*

$$\{\, p\text{-divisible formal group laws over } R \,\} \xrightarrow{\sim} \{\, \text{connected } p\text{-divisible groups over } R \,\}$$

which maps each p-divisible formal group law μ over R to $\mu[p^\infty]$.

PROOF. Let μ and ν be formal group laws of degree d over R. Let us define $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ and $B_v := \mathscr{A}/[p^v]_\nu(\mathscr{I})$. By Proposition 2.2.10 $\mu[p^v] := \mathrm{Spec}\,(A_v)$ and $\nu[p^v] := \mathrm{Spec}\,(B_v)$ are connected finite flat R-group scheme. Let μ_v and ν_v denote the comultiplications of $\mu[p^v]$ and $\nu[p^v]$. We write $\mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v)$ for the set of R-algebra maps $B_v \to A_v$ which are compatible with the comultiplications ν_v and μ_v, and $\mathrm{Hom}_{\nu,\mu}(\mathscr{A}, \mathscr{A})$ for the set of continuous ring homomorphisms $\mathscr{A} \to \mathscr{A}$ which are compatible with ν and μ in the sense of the commutative diagram in Definition 2.2.6. By Lemma 2.2.15 we have

$$\begin{aligned}\operatorname{Hom}(\mu,\nu) &= \operatorname{Hom}_{\nu,\mu}(\mathscr{A},\mathscr{A}) = \operatorname{Hom}_{\nu,\mu}(\varprojlim B_v, \varprojlim A_v)\\ &= \varprojlim \operatorname{Hom}_{\nu_v,\mu_v}(B_v, A_v) = \varinjlim \operatorname{Hom}_{R\text{-grp}}(\mu[p^v],\nu[p^v]) = \operatorname{Hom}(\mu[p^\infty],\nu[p^\infty]).\end{aligned}$$

We thus deduce that the functor is fully faithful.

Let $G = \varinjlim G_v$ be an arbitrary connected p-divisible group over R. We write $G_v = \operatorname{Spec}(A_v)$ where A_v is a free local R-algebra of finite rank. Let $p_v : A_{v+1} \to A_v$ denote the R-algebra homomorphism induced by the transition map $G_v \to G_{v+1}$. Note that each p_v is surjective as the corresponding transition map $G_v \to G_{v+1}$ is a closed embedding.

By Proposition 2.2.13 we have a continuous isomorphism

$$k[[t_1,\cdots,t_d]] \simeq \varprojlim (A_v \otimes_R k). \tag{2.5}$$

We aim to lift this isomorphism to a homomorphism

$$f : \mathscr{A} = R[[t_1,\cdots,t_d]] \to \varprojlim A_v.$$

In other words, we construct a lift $f_v : \mathscr{A} \to A_v$ of each projection $k[[t_1,\cdots,t_d]] \twoheadrightarrow A_v \otimes_R k$ so that the following diagram commutes:

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{f_{v+1}} & A_{v+1} & \longrightarrow & A_{v+1}\otimes_R k \\ & \searrow^{f_v} & \downarrow{\scriptstyle p_v} & & \downarrow{\scriptstyle p_v\otimes \mathrm{id}} \\ & & A_v & \longrightarrow & A_v \otimes_R k \end{array}$$

After taking f_1 to be any lift of the projection $k[[t_1,\cdots,t_d]] \twoheadrightarrow A_1 \otimes_R k$, we proceed by induction on v. Let us choose $y_1,\cdots,y_d \in A_{v+1}$ which lift the images of $t_1,\cdots,t_d$ under the projection $k[[t_1,\cdots,t_d]] \twoheadrightarrow A_{v+1}\otimes_R k$. Then $p_v(y_1),\cdots,p_v(y_d)$ should lift the images of $t_1,\cdots,t_d$ under the projection $k[[t_1,\cdots,t_d]] \twoheadrightarrow A_v \otimes_R k$. Since f_v is a lift of the projection $k[[t_1,\cdots,t_d]] \twoheadrightarrow A_v\otimes_R k$, we have $f_v(t_i) - p_v(y_i) \in \mathfrak{m}A_v$ where $\mathfrak{m}$ denotes the maximal ideal of R. By surjectivity of p_v, we may choose $z_1,\cdots,z_d \in \mathfrak{m}A_{v+1}$ with $p_v(z_i) = f_v(t_i) - p_v(y_i)$. Let us now define $f_{v+1} : \mathscr{A} \to A_{v+1}$ by setting $f_{v+1}(t_i) = y_i + z_i$. From our construction, we quickly verify that f_{v+1} is a desired lift of the projection $k[[t_1,\cdots,t_d]] \twoheadrightarrow A_{v+1}\otimes_R k$.

We assert that f is indeed an isomorphism. Nakayama's lemma implies surjectivity of each f_v, which in turn implies surjectivity of f. Moreover, we find $\varprojlim A_v \simeq R[[u]]$ as an R-module since each $p_v : A_{v+1} \twoheadrightarrow A_v$ admits an R-module splitting for being a surjective map between two free R-modules. Hence f splits in the sense of R-modules as well. It is also clear that this splitting is compatible with passage to the quotient modulo $\mathfrak{m}$. In particular, by the isomorphism (2.5) we have $\ker(f)\otimes_R k = 0$, or equivalently $\mathfrak{m}\ker(f) = \ker(f)$. Denoting by $\mathfrak{M}$ the maximal idea of $\mathscr{A}$, we find

$$\mathfrak{M}\ker(f) = (\mathfrak{m}\mathscr{A} + (t_1,\cdots,t_d))\ker(f) = \ker(f).$$

As $\mathscr{A} = R[[t_1,\cdots,t_d]]$ is noetherian, we deduce $\ker(f) = 0$ by Nakayama's lemma.

The formulation of f commutes with passage to quotients modulo $\mathfrak{m}^n$ for any n. Moreover, the kernels of the projections $\mathscr{A} \twoheadrightarrow A_v$ form a system of open ideals in $\mathscr{A}$ as the R-algebras A_v are of finite length. Hence by a theorem of Chevalley as stated in [**Mat87**, Exercise 8.7] we deduce that f is a continuous isomorphism.

Let us now denote the comultiplication of G_v by μ_v, and take $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ to be $\varprojlim \mu_v$ via the isomorphism f. The axioms for each comultiplication μ_v implies that μ fits in the commutative diagrams in (1) and (2) of Lemma 2.2.5, which in turn implies that μ is indeed a formal group law over R. Now let $\eta_{v,t} : A_t \hookrightarrow A_{v+t}$ denote the injective ring homomorphism induced by $j_{v,t} : G_{v+t} \twoheadrightarrow G_t$ as defined in Lemma 2.1.5. Since we have $[p^v]_G = \varinjlim j_{v,t}$, the isomorphism f yields an identification $[p^v]_\mu = \varprojlim \eta_{v,t}$. It is then straightforward to check that $[p]_\mu$ is finite flat, which means that μ is p-divisible. We then find $\mu[p^v] \cong G_v$ and $\mu[p^\infty] \cong G$, thereby establishing the essential surjectivity of the functor. □

Remark. The last paragraph of the proof can be simplified by considering the formal group $\mathscr{G}_\mu$ associated to μ. In fact, as it makes sense to write $\mathscr{G}_\mu \cong \varinjlim G_v$ as a formal scheme, we immediately obtain the identification $\mathscr{G}_\mu[p^v] \cong G_v$ and the p-divisibility of $\mathscr{G}_\mu$ by observing $[p^v]_G = \varinjlim j_{v,t}$. Then we complete the proof by identifying $\mu[p^v] \cong \mathscr{G}_\mu[p^v]$ for each v as remarked after Proposition 2.2.10.

Definition 2.2.17. Let G be a p-divisible group over R. We write $\mu(G)$ for the unique p-divisible formal group law over R with $\mu(G)[p^\infty] \simeq G^\circ$ as given by Theorem 2.2.16, and define the *dimension* of G to be the dimension of $\mu(G)$.

Corollary 2.2.18. *Let G be a p-divisible group over R. Let us write $\overline{G} := G \times_R k$. Then $\ker(\varphi_{\overline{G}})$ has order p^d where d is the dimension of G.*

Proof. Proposition 1.5.9 implies that $\ker(\varphi_{\overline{G}})$ lies in $\overline{G}^\circ := G^\circ \times_R k$. Hence the assertion follows from Proposition 2.2.13, Theorem 2.2.16, and their proofs. □

We finish this subsection by discussing several important applications of Theorem 2.2.16.

Theorem 2.2.19. *Let G be a p-divisible group of height h over R. Let d and $d^\vee$ denote the dimensions of G and $G^\vee$, respectively. Then we have $h = d + d^\vee$.*

Proof. Let us write $\overline{G} := G \times_R k$ and $\overline{G} = \varinjlim \overline{G}_v$ where each $\overline{G}_v$ is a finite flat k-group scheme. Note that $\ker(\varphi_{\overline{G}})$ must lie in $\overline{G}[p] \cong \overline{G}_1$ since $\psi_{\overline{G}} \circ \varphi_{\overline{G}} = [p]_{\overline{G}}$ by Proposition 2.1.12. In particular, we have $\ker(\varphi_{\overline{G}}) \cong \ker(\varphi_{\overline{G}_1})$. We similarly find $\ker(\psi_{\overline{G}}) \cong \ker(\psi_{\overline{G}_1})$.

Let us consider the diagram

$$\begin{array}{ccccccccc} \underline{0} & \longrightarrow & \ker(\varphi_{\overline{G}}) & \longrightarrow & \overline{G} & \xrightarrow{\varphi_{\overline{G}}} & \overline{G}^{(p)} & \longrightarrow & \underline{0} \\ & & \downarrow & & \downarrow{\scriptstyle [p]_{\overline{G}}} & & \downarrow{\scriptstyle \psi_{\overline{G}}} & & \\ \underline{0} & \longrightarrow & \underline{0} & \longrightarrow & \overline{G} & \xrightarrow{\mathrm{id}} & \overline{G} & \longrightarrow & \underline{0}. \end{array}$$

The left square commutes since $\ker(\varphi_{\overline{G}})$ must lie in $\overline{G}[p]$ as already noted, while the right square commutes by Proposition 2.1.12. In addition, the first row is exact since $\varphi_{\overline{G}}$ is surjective as noted in the proof of Proposition 2.2.13, while the second row is visibly exact. Hence by the snake lemma we obtain an exact sequence

$$\underline{0} \longrightarrow \ker(\varphi_{\overline{G}}) \longrightarrow \ker([p]_{\overline{G}}) \longrightarrow \ker(\psi_{\overline{G}}) \longrightarrow \underline{0}. \tag{2.6}$$

We now compute the order of $\ker(\psi_{\overline{G}}) \cong \ker(\psi_{\overline{G}_1})$. As $\psi_{\overline{G}_1} = \varphi_{\overline{G}_1^\vee}^\vee$ by definition, we may identify $\ker(\psi_{\overline{G}_1})$ with the cokernel of $\varphi_{G_1^\vee}$ by the exactness of Cartier duality. Moreover, since $\overline{G}_1^\vee$ and $(\overline{G}_1^\vee)^{(p)}$ have the same order, we use the mutiplicativity of orders in short exact sequences to find that the cokernel of $\varphi_{\overline{G}_1^\vee}$ has the same order as $\ker(\varphi_{\overline{G}_1^\vee}) \cong \ker(\varphi_{\overline{G}^\vee})$. We thus deduce from Corollary 2.2.18 that $\ker(\psi_{\overline{G}})$ has order $p^{d^\vee}$.

Note that $\ker(\varphi_{\overline{G}})$ has order p^d by Corollary 2.2.18. Since $\ker([p]_{\overline{G}}) \cong G_1$ has order p^h, the multiplicativity of orders in the exact sequence (2.6) yields $p^h = p^{d+d^\vee}$, or equivalently $h = d + d^\vee$ as desired. □

Proposition 2.2.20. *Assume that k is an algebraically closed field of characteristic p. Every p-divisible group of height 1 over k is isomorphic to either $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ or μ_{p^∞}.*

PROOF. Let $G = \varinjlim G_v$ be an étale p-divisible group of height h over k. Proposition 2.1.10 implies that G is either étale or connected.

Let us first consider the case where G is étale. Then each G_v is a finite étale R-group scheme of order p^{hv} such that $G_v = G_{v+1}[p^v]$. By Proposition 1.3.1, each $G_v(\overline{k})$ is an abelian group of order p^{hv} such that $G_v(\overline{k})$ is the p^v-torsion subgroup of $G_{v+1}(\overline{k})$. An easy induction shows $G_v(\overline{k}) \simeq \mathbb{Z}/p^v\mathbb{Z}$, which in turn implies $G_v \simeq \underline{\mathbb{Z}/p^v\mathbb{Z}}$ by Proposition 1.3.1. We thus find $G \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

Let us now turn to the case where G is connected. As G has dimension 1, Theorem 2.2.19 implies that $G^\vee$ is zero dimensional and thus étale. Hence by the discussion in the preceding paragraph we find $G^\vee \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ or equivalently $G \simeq (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^\vee \simeq \mu_{p^\infty}$. □

Remark. The argument in the second paragraph readily extends to show that every étale p-divisible group of height h over k is isomorphic to $\underline{\mathbb{Q}_p/\mathbb{Z}_p}^h$.

Example 2.2.21. Let E be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. By Proposition 2.1.10 we have an exact sequence

$$\underline{0} \longrightarrow E[p^\infty]^\circ \longrightarrow E[p^\infty] \longrightarrow E[p^\infty]^{\text{ét}} \longrightarrow \underline{0}.$$

Note that both $E[p^\infty]^\circ$ and $E[p^\infty]^{\text{ét}}$ are nontrivial since both $E[p]^\circ$ and $E[p]^{\text{ét}} := E[p]/E[p]^\circ$ are nontrivial as seen in Example 1.4.12. Since $E[p^\infty]$ has height 2, we deduce that $E[p^\infty]^\circ$ and $E[p^\infty]^{\text{ét}}$ both have height 1. Hence the above exact sequence becomes

$$\underline{0} \longrightarrow \mu_{p^\infty} \longrightarrow E[p^\infty] \longrightarrow \underline{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \underline{0}$$

by Proposition 2.2.20. Moreover, this exact sequence splits as it splits at every finite level by Proposition 1.4.11. We thus find

$$E[p^\infty] \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}.$$

Remark. Let us extend our discussion in Example 2.2.21 to describe the Serre-Tate deformation theory for ordinary elliptic curves. The general Serre-Tate deformation theory says that a deformation of an abelian variety over a perfect field of characteristic p is equivalent to a deformation of its p-divisible group. Hence the deformation theory of an ordinary elliptic curve E over $\overline{\mathbb{F}}_p$ is the same as the deformation theory for the p-divisible group $E[p^\infty] \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}$. Moreover, as our discussion in Example 1.4.12 equally applies for ordinary elliptic curves over any deformation ring, every deformation of E should be an extension of $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ by μ_{p^∞}. We thus find that the deformation space of E is naturally isomorphic to $\underline{\mathrm{Ext}}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty})$. Furthermore, by the short exact sequence

$$\underline{0} \longrightarrow \underline{\mathbb{Z}_p} \longrightarrow \underline{\mathbb{Q}_p} \longrightarrow \underline{\mathbb{Q}_p/\mathbb{Z}_p} \longrightarrow \underline{0}$$

we obtain an identification $\underline{\mathrm{Ext}}^1(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, \mu_{p^\infty}) \cong \underline{\mathrm{Hom}}(\underline{\mathbb{Z}_p}, \mu_{p^\infty})$, which has the natural structure of a formal torus of dimension 1 as described in Example 2.2.4. The unit section corresponds to a unique deformation of E, called the *canonical deformation* of E, for which the exact sequence as described in Example 1.4.12 splits. The canonical deformation is also a unique deformation of E which lifts all endomorphisms of E.

2.3. Dieudonné-Manin classification

Our main goal in this subsection is to introduce two classes of semilinear algebraic objects that are closely related to p-divisible groups. We begin by recalling without proof that the ring of Witt vectors over a perfect $\mathbb{F}_p$-algebra satisfies the following universal property:

Lemma 2.3.1. *Let A be a perfect $\mathbb{F}_p$-algebra, and let R be a p-adically complete ring. Denote by $W(A)$ the ring of Witt vectors over A. Let $\overline{\pi} : A \longrightarrow R/pR$ be a ring homomorphism. Then $\overline{\pi}$ uniquely lifts to a multiplicative map $\hat{\pi} : A \longrightarrow R$ and a ring homomorphism $\pi : W(A) \longrightarrow R$. In addition, we have*

$$\pi\left(\sum_{n=0}^{\infty}[a_n]p^n\right) = \sum_{n=0}^{\infty}\hat{\pi}(a_n)p^n \qquad \text{for every } a_n \in A$$

where $[a_n]$ denotes the Teichmüler lift of a_n in $W(A)$.

Remark. For a proof, we refer the readers to [**Ked15**, Lemma 1.1.6].

Example 2.3.2. Let $W(A)$ be the ring of Witt vectors over a perfect $\mathbb{F}_p$-algebra A. By Lemma 2.3.1 the p-th power map on A uniquely lifts to an endomorphism $\varphi_{W(A)}$ on $W(A)$, called the *Frobenius automorphism* of $W(A)$, which satisfies

$$\varphi_{W(A)}\left(\sum_{n=0}^{\infty}[a_n]p^n\right) = \sum_{n=0}^{\infty}[a_n^p]p^n \qquad \text{for all } a_n \in A$$

where $[a_n]$ and $[a_n^p]$ respectively denote the Teichmüler lifts of a_n and a_n^p in $W(A)$. The perfectness of A implies that $\varphi_{W(A)}$ is indeed an automorphism.

Remark. For $A = \mathbb{F}_q$, we have identification $W(\mathbb{F}_q) \cong \mathbb{Z}_p[\zeta_{q-1}]$ where ζ_{q-1} denotes a primitive $(q-1)$-st root of unity. Then the Frobenius automorphism $\sigma_{W(\mathbb{F}_q)}$ sends ζ_{q-1} to ζ_{q-1}^p while acting trivially on $\mathbb{Z}_p$.

For the rest of this section, we let k be a perfect field of characteristic p. We also write $W(k)$ for the ring of Witt vectors over k, and $K_0(k)$ for the fraction field of $W(k)$.

Definition 2.3.3. Let σ denote the Frobenius automorphism of $W(k)$.

(1) We define the *Frobenius automorphism* of $K_0(k)$ to be the unique field automorphism on $K_0(k)$ which extends σ.
(2) Given two $W(k)$-modules M and N, we say that an additive map $f : M \longrightarrow N$ is *σ-semilinear* if it satisfies
$$f(am) = \sigma_{W(k)}(a)f(m) \qquad \text{for all } a \in W(k) \text{ and } m \in M.$$
(3) A *Dieudonné module of rank r* over k is a free $W(k)$-module M of rank r with a σ-semilinear endomorphism φ_M, called the *Frobenius endomorphism* of M, whose image contains pM.
(4) An *isocrystal of rank r* over $K_0(k)$ is an r-dimensional $K_0(k)$-space N with a σ-semilinear automorphism φ_N called the *Frobenius automorphism* of N.
(5) Given two Dieudonné modules M_1 and M_2 over k, a $W(k)$-linear map $f : M_1 \longrightarrow M_2$ is called a *morphism of Dieudonné modules* if it satisfies
$$f(\varphi_{M_1}(m)) = \varphi_{M_2}(f(m)) \qquad \text{for all } m \in M_1.$$
(6) Given two isocrystals N_1 and N_2 over $K_0(k)$, a $K_0(k)$-linear map $g : N_1 \longrightarrow N_2$ is called a *morphism of isocrystals* if it satisfies
$$g(\varphi_{N_1}(n)) = \varphi_{N_2}(g(n)) \qquad \text{for all } n \in N_1.$$

Lemma 2.3.4. *Let σ denote the Frobenius automorphism of $K_0(k)$.*

(1) *Every Dieudonné module M over k naturally gives rise to an isocrystal $M[1/p] = M \otimes_{W(k)} K_0(k)$ over $K_0(k)$ with the Frobenius automorphism $\varphi_M \otimes 1$.*
(2) *Given an isocrystal N over $K_0(k)$, the dual space $N^\vee = \mathrm{Hom}_{K_0(k)}(N, K_0(k))$ is naturally an isocrystal over $K_0(k)$ with the Frobenius automorphism $\varphi_{N^\vee}$ given by*
$$\varphi_{N^\vee}(f)(n) = \sigma(f(\varphi_N^{-1}(n))) \qquad \text{for all } f \in N^\vee \text{ and } n \in N.$$
(3) *Given two isocrystals N_1 and N_2 over $K_0(k)$, the vector space $N_1 \otimes_{K_0(k)} N_2$ is naturally an isocrystal over $K_0(k)$ with the Frobenius automorphism $\varphi_{N_1} \otimes \varphi_{N_2}$.*

Proof. All statements are straightforward to verify using Definition 2.3.3. □

Remark. The category of Dieudonné modules over k also admits a natural notion of tensor product and dual.

Example 2.3.5. Let N be an isocrystal of rank r over $K_0(k)$. Lemma 2.3.4 implies that $\det(N) = \wedge^r(N)$ is naturally an isocrystal of rank 1 over $K_0(k)$, which we refer to as the *determinant* of N.

We now introduce several fundamental theorems that allow us to study p-divisible groups and abelian varieties over k using semilinear algebraic objects defined in Definition 2.3.3. We won't provide their proofs, as we will only use these theorems as motivations for some key constructions in Chapter III and IV. The readers may find an excellent exposition of these theorems in [**Dem72**, Chapters III and IV].

Theorem 2.3.6 (Dieudonné [**Die55**])**.** *There is an exact anti-equivalence of categories*
$$\mathbb{D} : \{\, p\text{-divisible groups over } k \,\} \xrightarrow{\sim} \{\, \text{Dieudonné modules over } k \,\}$$
such that for an arbitrary p-divisible group G over k we have the following statements:

(1) *The rank of $\mathbb{D}(G)$ is equal to the height of G.*
(2) *G is étale if and only if $\varphi_{\mathbb{D}(G)}$ is bijective.*
(3) *G is connected if and only if $\varphi_{\mathbb{D}(G)}$ is topologically nilpotent.*
(4) *$[p]_G$ induces the multiplication by p on $\mathbb{D}(G)$.*
(5) *There exists a canonical identification $\mathbb{D}(G^\vee)[1/p] \cong \mathbb{D}(G)[1/p]^\vee$.*

Definition 2.3.7. We refer to the functor $\mathbb{D}$ described in Theorem 2.3.6 as the *Dieudonné functor.*

Example 2.3.8. Let σ denote the Frobenius automorphism of $W(k)$.

(1) $\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})$ is isomorphic to $W(k)$ together with $\varphi_{\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})} = \sigma$.
(2) $\mathbb{D}(\mu_{p^\infty})$ is isomorphic to $W(k)$ together with $\varphi_{\mathbb{D}(\mu_{p^\infty})} = p\sigma$.
(3) If E is an ordinary elliptic curve over $\overline{k}$, we have $\mathbb{D}(E[p^\infty]) \simeq W(\overline{k})^{\oplus 2}$ together with $\varphi_{\mathbb{D}(E[p^\infty])} = \sigma \oplus p\sigma$.

Remark. We can also define the *Verschiebung endomorphism* ψ on $W(k)$ which satisfies
$$\psi\left(\sum_{n=0}^{\infty}[a_n]p^n\right) = \sum_{n=1}^{\infty}[a_{n-1}]p^n \qquad \text{for all } a_n \in k$$
where $[a_n]$ denotes the Teichmüler lift of a_n in $W(k)$. It is then straightforward to check that $\sigma \circ \psi$ and $\psi \circ \sigma$ are both equal to the multiplication by p on $W(k)$. Hence we can recover Proposition 1.5.8 by applying Theorem 2.3.6 to the first example above.

Definition 2.3.9. A homomorphism $f : G \to H$ of p-divisible groups over k is called an *isogeny* if it is surjective (as a map of fpqc sheaves) with finite flat kernel.

Example 2.3.10. We present some important examples of isogenies between p-divisible groups.

(1) Given a p-divisible group G over k, the homomorphisms $[p]_G, \varphi_G$, and ψ_G are all isogenies.
(2) An isogeny $A \to B$ of two abelian varieties over k induces an isogeny $A[p^\infty] \to B[p^\infty]$.

Proposition 2.3.11. *A homomorphism $f : G \to H$ of p-divisible groups over k is an isogeny if and only if the following equivalent conditions are satisfied.*

(i) The induced map $\mathbb{D}(H) \to \mathbb{D}(G)$ is injective.
(ii) The induced map $\mathbb{D}(H)[1/p] \to \mathbb{D}(G)[1/p]$ is an isomorphism.

Corollary 2.3.12. *Let G be a p-divisible group over k. The isogeny class of G is determined by the isomorphism class of the isocrystal $\mathbb{D}(G)[1/p]$.*

Definition 2.3.13. Let N be an isocrystal of rank r over $K_0(\overline{k})$.

(1) The *degree* of N is the largest integer $\deg(N)$ with $\varphi_{\det(N)}(1) \in p^{\deg(N)}W(k)$, where $\varphi_{\det(N)}$ denotes the Frobenius automorphism of $\det(N)$.
(2) We write $\mathrm{rk}(N)$ for the rank of N, and define the *slope* of N by $\mu(N) := \dfrac{\deg(N)}{\mathrm{rk}(N)}$.

Example 2.3.14. Let $\lambda = d/r$ be a rational number written in lowest terms with $r > 0$. The *simple isocrystal of slope* λ over $K_0(k)$, denoted by $N(\lambda)$, is the $K_0(k)$-space $K_0(k)^{\oplus r}$ together with the σ-semilinear automorphism $\varphi_{N(\lambda)}$ given by

$$\varphi_{N(\lambda)}(e_1) = e_2, \cdots, \varphi_{N(\lambda)}(e_{r-1}) = e_r, \varphi_{N(\lambda)}(e_r) = p^d e_1,$$

where $e_1, \cdots, e_r$ denote the standard basis vectors. It is straightforward to verify that $N(\lambda)$ is of rank r, degree d, and slope λ.

Theorem 2.3.15 (Manin [**Man63**]). *Every isocrystal N over $K_0(\overline{k})$ admits a unique direct sum decomposition of the form*

$$N \simeq \bigoplus_{i=1}^{l} N(\lambda_i)^{\oplus m_i}$$

for some $\lambda_i \in \mathbb{Q}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_l$.

Definition 2.3.16. Let N be an isocrystal over $K_0(\overline{k})$ with a direct sum decomposition

$$N \simeq \bigoplus_{i=1}^{l} N(\lambda_i)^{\oplus m_i}$$

for some $\lambda_i \in \mathbb{Q}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_l$. For each i, let us write $\lambda_i = d_i/r_i$ for the lowest form with $r_i > 0$.

(1) Each λ_i is called a *Newton slope* of N with multiplicity m_i.
(2) The *Newton polygon* of N, denoted by $\mathrm{Newt}(N)$, is the lower convex hull of the points $(0, 0)$ and $(m_1r_1 + \cdots + m_ir_i, m_1d_1 + \cdots + m_id_i)$ in $\mathbb{R}^2$.

Example 2.3.17. For an ordinary elliptic curve E over $\overline{k}$, we have an isomorphism

$$\mathbb{D}(E[p^\infty])[1/p] \simeq N(0) \oplus N(1).$$

The Newton polygon of $\mathbb{D}(E[p^\infty])[1/p]$ connects the points $(0, 0), (1, 0)$, and $(2, 1)$.

Proposition 2.3.18. *Let N be an isocrystal over $K_0(\overline{k})$. Then we have $N \simeq \mathbb{D}(G)[1/p]$ for some p-divisible group G of height h and dimension d over $\overline{k}$ if and only if the following conditions are satisfied:*

(i) *N is of rank h and degree d.*
(ii) *Every Newton slope λ of N satisfies $0 \leq \lambda \leq 1$.*

Theorem 2.3.19 (Serre-Honda-Tate [**Tat71**], Oort [**Oor00**]). *Let N be an isocrystal over $K_0(\overline{k})$. There exists a principally polarized abelian variety A of dimension g over $\overline{k}$ with $N \simeq \mathbb{D}(A[p^\infty])[1/p]$ if and only if the following conditions are satisfied:*

(i) *N is of rank $2g$ and degree g.*
(ii) *Every Newton slope λ of N satisfies $0 \leq \lambda \leq 1$.*
(iii) *If $\lambda \in \mathbb{Q}$ occurs as a Newton slope of N, then $1 - \lambda$ occurs as a Newton slope of N with the same multiplicity.*

Remark. The necessity part is easy to verify by Proposition 2.3.18. The main difficulty lies in proving the sufficiency part, which was initially conjectured by Manin [**Man63**].

Definition 2.3.20. Let A be a principally polarized abelian variety of dimension g over $\overline{k}$.

(1) We define its Newton polygon by $\mathrm{Newt}(A) := \mathrm{Newt}(\mathbb{D}(A[p^\infty])[1/p])$.
(2) We say that A is *ordinary* if $\mathrm{Newt}(A)$ connects the points $(0,0), (g,0)$, and $(2g,g)$.
(3) We say that A is *supersingular* if $\mathrm{Newt}(A)$ connects the points $(0,0)$ and $(2g,g)$.

Example 2.3.21. Let A be an ordinary abelian variety of dimension g over $\overline{k}$. A priori, this means that there exists an isogeny $A[p^\infty] \to \mu_{p^\infty}^g \times (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g$. We assert that there exists an isomorphism

$$A[p^\infty] \simeq \mu_{p^\infty}^g \times (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g.$$

By Proposition 2.1.10 we have an exact sequence

$$\underline{0} \longrightarrow A[p^\infty]^\circ \longrightarrow A[p^\infty] \longrightarrow A[p^\infty]^{\text{ét}} \longrightarrow \underline{0}.$$

Moreover, this sequence splits as it splits at every finite level by Proposition 1.4.11. Hence we have a decomposition

$$A[p^\infty] \simeq A[p^\infty]^\circ \times A[p^\infty]^{\text{ét}}.$$

Proposition 2.3.18 implies that $A[p^\infty]^{\text{ét}}$ should correspond to the slope 0 part of $\mathrm{Newt}(A)$, and thus have height g. We then deduce $A[p^\infty]^{\text{ét}} \simeq (\underline{\mathbb{Q}_p/\mathbb{Z}_p})^g$ by the remark after Proposition 2.2.20, and $A[p^\infty]^\circ \simeq (A[p^\infty]^{\text{ét}})^\vee \simeq \mu_{p^\infty}^g$ by self-duality of $A[p^\infty]$.

Remark. We can also argue as in the remark after Example 2.2.21 to deduce that the deformation space of A has the structure of a formal torus of dimension $g(g+1)/2$.

Proposition 2.3.22. *For an abelian variety A over k, there exist a natural identification*

$$H^1_{\mathrm{cris}}(A/W(k)) \cong \mathbb{D}(A[p^\infty]).$$

Remark. In light of the crystalline comparison theorem as introduced in Chapter I, Theorem 1.2.4, this identification provides a powerful tool to study abelian varieties and their moduli spaces, such as (local) Shimura varieties of PEL or Hodge type, using p-adic Hodge theory and the theory of Dieudonné modules/isocrystals.

3. Hodge-Tate decomposition

In this section, we finally enter the realm of p-adic Hodge theory. Assuming some technical results from algebraic number theory, we prove two fundamental theorems regarding p-divisible groups, namely the Hodge-Tate decomposition for the Tate modules and the full-faithfulness of the generic fiber functor. The primary reference for this section is Tate's paper [**Tat67**].

3.1. The completed algebraic closure of a p-adic field

Definition 3.1.1. Let K be an extension of $\mathbb{Q}_p$ with a nonarchimedean valuation ν.

(1) We define the *valuation ring* of K by $\mathcal{O}_K := \{ x \in K : \nu(x) \geq 0 \}$.
(2) We say that K is a *p-adic field* if it is discrete valued and complete with a perfect residue field.

Example 3.1.2. We present some essential examples of p-adic fields.

(1) Every finite extension of $\mathbb{Q}_p$ is a p-adic field.
(2) Given a perfect field k of characteristic p, the fraction field of the ring of Witt vectors $W(k)$ is a p-adic field.

Remark. The fraction field of $W(\overline{\mathbb{F}}_p)$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$. Hence it is a p-adic field which is not an algebraic extension of $\mathbb{Q}_p$.

For the rest of this section, we let K be a p-adic field with absolute Galois group Γ_K. We also write $\mathfrak{m}$ and k for the maximal ideal and the residue field of $\mathcal{O}_K$.

Definition 3.1.3. We define the *completed algebraic closure* of K by $\mathbb{C}_K := \widehat{\overline{K}}$; in other words, $\mathbb{C}_K$ is the p-adic completion of the algebraic closure of K.

Remark. The field $\mathbb{C}_K$ is not a p-adic field as its valuation is not discrete. In fact, it is the first example of a characteristic 0 perfectoid field.

Lemma 3.1.4. *The action of Γ_K on K uniquely extends to a continuous action on $\mathbb{C}_K$.*

Proof. This is obvious by continuity of the Γ_K-action on K. □

For the rest of this section, we fix a valuation ν on $\mathbb{C}_K$ with $\nu(p) = 1$.

Proposition 3.1.5. *The field $\mathbb{C}_K$ is algebraically closed.*

Proof. Let $p(t)$ be an arbitrary non-constant polynomial over $\mathbb{C}_K$. We wish to prove that $p(t)$ has a root in $\mathbb{C}_K$. By scaling the variable if necessary, we may assume that $p(t)$ is a monic polynomial over $\mathcal{O}_{\mathbb{C}_K}$. In other words, we may write

$$p(t) = t^d + a_1 t^{d-1} + \cdots + a_d$$

for some $a_i \in \mathcal{O}_{\mathbb{C}_K}$. For each n, we choose a polynomial

$$p_n(t) = t^d + a_{1,n} t^{d-1} + \cdots + a_{d,n}$$

with $a_{i,n} \in \mathcal{O}_{\overline{K}}$ and $\nu(a_i - a_{i,n}) \geq dn$.

Let us choose $\alpha_1 \in \mathcal{O}_{\overline{K}}$ with $p_1(\alpha_1) = 0$. We proceed by induction on n to choose $\alpha_n \in \mathcal{O}_{\overline{K}}$ with $p_n(\alpha_n) = 0$ and $\nu(\alpha_n - \alpha_{n-1}) \geq n-1$. Since $a_{i,n} - a_{i,n-1} = (a_{i,n} - a_i) + (a_i - a_{i,n-1})$ has valuation at least $d(n-1)$, we find $\nu(p_n(\alpha_{n-1})) \geq d(n-1)$ by observing

$$p_n(\alpha_{n-1}) = p_n(\alpha_{n-1}) - p_{n-1}(\alpha_{n-1}) = \sum_{i=1}^{d} (a_{i,n} - a_{i,n-1}) \alpha_{n-1}^{d-i}.$$

Moreover, we have

$$p_n(\alpha_{n-1}) = \prod_{i=1}^{d}(\alpha_{n-1} - \beta_{n,i})$$

where $\beta_{n,1}, \cdots, \beta_{n,d}$ are roots of $p_n(t)$. Note that $\beta_{n,i} \in \mathcal{O}_{\overline{K}}$ since $\mathcal{O}_{\overline{K}}$ is integrally closed. As $\nu(p_n(\alpha_{n-1})) \geq d(n-1)$, we deduce that $\nu(\alpha_{n-1} - \beta_{n,i}) \geq n-1$ for some i. We thus complete the induction step by taking $\alpha_n := \beta_{n,i}$.

Since the sequence (α_n) is Cauchy by construction, it converges to an element $\alpha \in \mathcal{O}_{\mathbb{C}_K}$. Moreover, for each n we find $\nu(p(\alpha_n)) \geq dn$ by observing

$$p(\alpha_n) = p(\alpha_n) - p_n(\alpha_n) = \sum_{i=1}^{d}(a_i - a_{i,n})\alpha_n^{d-i}.$$

We thus have $p(\alpha) = 0$, thereby completing the proof. □

Let us now introduce the central objects for this course.

Definition 3.1.6. A *p-adic representation* of Γ_K is a finite dimensional $\mathbb{Q}_p$-vector space V together with a continuous homomorphism $\Gamma_K \to \mathrm{GL}(V)$. We denote by $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ the category of p-adic Γ_K-representations.

Example 3.1.7. Below are two essential examples of p-adic representations.

(1) By Proposition 2.1.14, every p-divisible group G over K gives rise to a p-adic Γ_K-representation $V_p(G) := T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, called the *rational Tate module* of G.
(2) For an arbitrary variety X over K, the étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is a p-adic Γ_K-representation.

Our main task in this section is to understand the p-adic Γ_K-representation on the rational Tate module of a p-divisible group over K. We will make extensive use of the following notion:

Definition 3.1.8. Given a $\mathbb{Z}_p[\Gamma_K]$-module M, we define its *n-th Tate twist* to be the $\mathbb{Z}_p[\Gamma_K]$-module

$$M(n) := \begin{cases} M \otimes_{\mathbb{Z}_p} T_p(\mu_{p^\infty})^{\otimes n} & \text{for } n \geq 0, \\ \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(\mu_{p^\infty})^{\otimes -n}, M) & \text{for } n < 0. \end{cases}$$

Example 3.1.9. By definition, we have $\mathbb{Z}_p(1) = T_p(\mu_{p^\infty}) = \varprojlim \mu_{p^v}(\overline{K})$. The homomorphism $\chi_K : \Gamma_K \to \mathrm{Aut}(\mathbb{Z}_p(1)) \cong \mathbb{Z}_p^\times$ which represents the Γ_K-action on $\mathbb{Z}_p(1)$ is called the *p-adic cyclotomic character* of K. We will often simply write χ instead of χ_K to ease the notation.

Lemma 3.1.10. *Let M be a $\mathbb{Z}_p[\Gamma_K]$-module. For each $m, n \in \mathbb{Z}$, we have canonical Γ_K-equivariant isomorphisms*

$$M(m) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \cong M(m+n) \qquad \textit{and} \qquad M(n)^\vee \cong M^\vee(-n).$$

PROOF. This is straightforward to check by definition. □

Lemma 3.1.11. *Let M be a $\mathbb{Z}_p[\Gamma_K]$-module, and let $\rho : \Gamma_K \to \mathrm{Aut}(M)$ be the homomorphism which represents the action of Γ_K on M. For every $n \in \mathbb{Z}$ the action of Γ_K on $M(n)$ is represented by $\chi^n \cdot \rho$.*

PROOF. Upon choosing a basis element e of $\mathbb{Z}_p(n)$, we obtain an isomorphism $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \xrightarrow{\sim} M$ given by $m \otimes e \mapsto m$. The assertion now follows by observing that the Γ_K-action on $M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n)$ is given by $\rho \otimes \chi^n$. □

We assume the following fundamental result about the Galois cohomology of the Tate twists of $\mathbb{C}_K$.

Theorem 3.1.12 (Tate [**Tat67**], Sen [**Sen80**]). *We have canonical isomorphisms*

$$H^i(\Gamma_K, \mathbb{C}_K(n)) \cong \begin{cases} K & \text{if } i = 0 \text{ or } 1, n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark. The proof of this result requires the full power of the higher ramification theory as well as some knowledge about the local class field theory. We refer curious readers to [**BC**, §14] for a complete proof.

When $i = n = 0$, the theorem says that the fixed field of Γ_K in $\mathbb{C}_K$ is K. This particular statement has an elementary proof as sketched in [**BC**, Proposition 2.1.2].

We now introduce the first class of p-adic Γ_K-representations.

Lemma 3.1.13 (Serre-Tate). *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, the natural $\mathbb{C}_K$-linear map*

$$\tilde{\alpha}_V : \bigoplus_{n\in\mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

is Γ_K-equivariant and injective.

Proof. For each $n \in \mathbb{Z}$, we have a Γ_K-equivariant K-linear map

$$\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K K(n) \hookrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n) \otimes_K K(n) \simeq V \otimes_{\mathbb{Q}_p} \mathbb{C}_K, \tag{3.1}$$

which gives rise to a Γ_K-equivariant $\mathbb{C}_K$-linear map

$$\tilde{\alpha}_V^{(n)} : \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

by extension of scalars. Hence we deduce that $\tilde{\alpha}_V = \bigoplus_{n\in\mathbb{Z}} \tilde{\alpha}_V^{(n)}$ is Γ_K-equivariant.

For each $n \in \mathbb{Z}$, we choose a basis $(v_{m,n})$ of $\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K K(n)$ over K. We may regard $v_{m,n}$ as a vector in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ via the map (3.1). Moreover, the source of the map $\tilde{\alpha}_V$ is spanned by the vectors $(v_{m,n})$.

Assume for contradiction that the kernel of $\tilde{\alpha}_V$ is not trivial. Then we have a nontrivial relation of the form $\sum c_{m,n} v_{m,n} = 0$. Let us choose such a relation with minimal length. We may assume that $c_{m_0,n_0} = 1$ for some m_0 and n_0. For every $\gamma \in \Gamma_K$ we find

$$0 = \gamma\left(\sum c_{m,n} v_{m,n}\right) - \chi(\gamma)^{n_0}\left(\sum c_{m,n} v_{m,n}\right) = \sum \left(\gamma(c_{m,n})\chi(\gamma)^n - \chi(\gamma)^{n_0} c_{m,n}\right) v_{m,n}$$

by Γ_K-equivariance of $\tilde{\alpha}_V$ and Lemma 3.1.11. Note that the coefficient of v_{m_0,n_0} in the last expression is 0. Hence the minimality of our relation implies that all coefficients in the last expression must vanish, thereby yielding relations

$$\gamma(c_{m,n})\chi(\gamma)^{n-n_0} = c_{m,n} \qquad \text{for all } \gamma \in \Gamma_K.$$

Then by Lemma 3.1.11 and Theorem 3.1.12 we find $c_{m,n} = 0$ for $n \neq n_0$ and $c_{m,n} \in K$ for $n = n_0$. Therefore our relation $\sum c_{m,n} v_{m,n} = 0$ becomes a nontrivial K-linear relation among the vectors v_{m,n_0}, thereby yielding a desired contradiction. □

Definition 3.1.14. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *Hodge-Tate* if the map $\tilde{\alpha}_V$ in Lemma 3.1.13 is an isomorphism.

Remark. We will see in §3.4 that p-adic representations discussed in Example 3.1.7 are Hodge-Tate in many cases.

3.2. Formal points on p-divisible groups

For the rest of this section, we fix the base ring $R = \mathcal{O}_K$. We also let L be the p-adic completion of an algebraic extension of K, and denote by $\mathfrak{m}_L$ its maximal ideal. We are particularly interested in the case where $L = \mathbb{C}_K$.

We investigate the notion of formal points on p-divisible groups over $\mathcal{O}_K$.

Definition 3.2.1. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. We define the *group of $\mathcal{O}_L$-valued formal points* on G by

$$G(\mathcal{O}_L) := \varprojlim_i G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) = \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

Example 3.2.2. By definition, $\mu_{p^\infty}(\mathcal{O}_L) = \varprojlim_i \mu_{p^\infty}(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ is the group of elements $x \in \mathcal{O}_L^\times$ such that $\nu(x^{p^v} - 1)$ can get arbitrarily large. Hence we clearly have $1 + \mathfrak{m}_L \subseteq \mu_{p^\infty}(\mathcal{O}_L)$. Moreover, as the residue field of $\mathcal{O}_L$ has characteristic p, we also obtain the opposite inclusion by observing $x^{p^v} - 1 = (x-1)^{p^v} \bmod \mathfrak{m}_L$. We thus find $\mu_{p^\infty}(\mathcal{O}_L) \cong 1 + \mathfrak{m}_L$.

Remark. On the other hand, the group of "ordinary" $\mathcal{O}_L$-valued points on μ_{p^∞} is given by

$$\varinjlim_v \mu_{p^v}(\mathcal{O}_L) = \varinjlim_v \left\{ x \in \mathcal{O}_L^\times : x^{p^v} = 1 \right\}$$

which precisely consists of p-power torsion elements in $\mathcal{O}_L^\times$. We thus see that $\mu_{p^\infty}(\mathcal{O}_L)$ contains many "non-ordinary" points.

Proposition 3.2.3. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$.*

(1) Writing $G_v = \mathrm{Spec}\,(A_v)$ for each v, we have an identification

$$G(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim_v A_v, \mathcal{O}_L).$$

(2) $G(\mathcal{O}_L)$ is a $\mathbb{Z}_p$-module with the torsion part given by

$$G(\mathcal{O}_L)_{\mathrm{tors}} \cong \varinjlim_v \varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

(3) If G is étale, then $G(\mathcal{O}_L)$ is isomorphic to a torsion group $G(k_L)$ where k_L denotes the residue field of $\mathcal{O}_L$

Proof. Note that we have $\mathcal{O}_L = \varprojlim_i \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ by completeness of $\mathcal{O}_L$. We also have $\varprojlim_v A_v = \varprojlim_{i,v} A_v/\mathfrak{m}^i A_v$ since each A_v is $\mathfrak{m}$-adically complete for being finite free over $\mathcal{O}_K$ by a general fact as stated in [**Sta**, Tag 031B]. We thus obtain an identification

$$\begin{aligned}
G(\mathcal{O}_L) &\cong \varprojlim_i \varinjlim_v \mathrm{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \varprojlim_i \varinjlim_v \mathrm{Hom}_{\mathcal{O}_K}(A_v/\mathfrak{m}^i A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\
&\cong \varprojlim_i \mathrm{Hom}_{\mathcal{O}_K}(\varprojlim_v A_v/\mathfrak{m}^i A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\
&\cong \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim_{i,v} A_v/\mathfrak{m}^i A_v, \varprojlim_i \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \\
&\cong \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim_v A_v, \mathcal{O}_L)
\end{aligned}$$

as asserted in (1).

Next we consider the statement (2). Observe that $G(\mathcal{O}_L)$ is a $\mathbb{Z}_p$-module since each $G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) = \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ is a $\mathbb{Z}_p$-module by Corollary 2.1.6. Hence $G(\mathcal{O}_L)_{\text{tors}}$ only contains p-power torsions. Moreover, by Corollary 2.1.6 we have an exact sequence

$$\underline{0} \longrightarrow G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \xrightarrow{[p^v]} G(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L),$$

which in turn yields an exact sequence

$$\underline{0} \longrightarrow \varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L) \xrightarrow{[p^v]} G(\mathcal{O}_L).$$

We find that the p^v-torsion part of $G(\mathcal{O}_L)$ is given by $\varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$, thereby deducing the assertion (2) by taking the limit over v.

If G is étale, we have identifications $G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong G_v(\mathcal{O}_L/\mathfrak{m}^{i+1}\mathcal{O}_L)$ by formal étaleness of étale morphisms as stated in [**Sta**, Tag 04AL], thereby obtaining

$$G(\mathcal{O}_L) = \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \varprojlim_i \varinjlim_v G_v(k_L) \cong G(k_L).$$

We thus deduce the statement (3) by Corollary 2.1.6. □

Remark. Arguing as in the proof of Theorem 2.2.16, we can show that the formal scheme $\mathscr{G} := \text{Spf}(\varprojlim A_v)$ carries the structure of a formal group induced by the finite flat $\mathcal{O}_K$-group schemes G_v. Moreover, we can write the identification in (1) as $G(\mathcal{O}_L) \cong \text{Hom}_{\mathcal{O}_K\text{-formal}}(\text{Spf}(\mathcal{O}_L), \mathscr{G})$.

Corollary 3.2.4. *Let G be a connected p-divisible group dimension d over $\mathcal{O}_K$. We have a canonical isomorphism of $\mathbb{Z}_p$-modules*

$$G(\mathcal{O}_L) \cong \text{Hom}_{\mathcal{O}_K-\text{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$$

where the multiplication by p on the target is induced by $[p]_{\mu(G)}$.

Remark. From the above isomorphism we obtain an identification $G(\mathcal{O}_L) \simeq \mathfrak{m}_{\mathcal{O}_L}^d$ as a set. It is then straightforward to check that μ induces the structure of a p-adic analytic group over L on $\mathfrak{m}_{\mathcal{O}_L}^d$ by Lemma 2.2.5 and the completeness of L.

Proposition 3.2.5. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. Then we have an exact sequence*

$$0 \longrightarrow G^\circ(\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L) \longrightarrow G^{\acute{e}t}(\mathcal{O}_L) \longrightarrow 0.$$

PROOF. Let us write $G^\circ = \varinjlim G^\circ$ and $G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$ where $G_v^{\text{ét}} := G_v/G_v^\circ$. We also write $G_v = \text{Spec}\,(A_v), G_v^\circ = \text{Spec}\,(A_v^\circ)$, and $G_v^{\text{ét}} = \text{Spec}\,(A_v^{\text{ét}})$ where A_v, A_v°, and $A_v^{\text{ét}}$ are finite free $\mathcal{O}_K$-algebras. In addition, we define $\mathscr{A} := \varprojlim A_v$ and $\mathscr{A}^{\text{ét}} := \varprojlim A_v^{\text{ét}}$.

Proposition 2.1.10 yields an exact sequence

$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow \underline{0}. \tag{3.2}$$

We wish to show that the induced sequence on the groups of $\mathcal{O}_L$-valued points is exact. The sequence is left exact by construction as limits and colimits are left exact in the category of abelian groups. Hence it remains to show surjectivity of the map $G(\mathcal{O}_L) \to G^{\text{ét}}(\mathcal{O}_L)$. By Proposition 3.2.3, it suffices to prove surjectivity of the map

$$\text{Hom}_{\mathcal{O}_K-\text{cont}}(\mathscr{A}, \mathcal{O}_L) \to \text{Hom}_{\mathcal{O}_K-\text{cont}}(\mathscr{A}^{\text{ét}}, \mathcal{O}_L). \tag{3.3}$$

By the proof of Theorem 2.2.16 we have a continuous isomorphism

$$\varprojlim A_v^\circ \simeq \mathcal{O}_K[[t_1, \cdots, t_d]]$$

where d is the dimension of G. Moreover, as the sequence (3.2) canonically splits after reduction to k by Proposition 1.4.11, we obtain a continuous isomorphism

$$(\mathscr{A}^{\text{ét}} \otimes_{\mathcal{O}_K} k)[[t_1, \cdots, t_d]] \simeq \mathscr{A} \otimes_{\mathcal{O}_K} k.$$

Arguing as in the proof of Theorem 2.2.16, we can lift the above map to a continuous homomorphism

$$f : \mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]] \to \mathscr{A}.$$

We assert that f is surjective. Assume for contradiction that $\operatorname{coker}(f) \neq 0$. Let $\mathfrak{M}$ be a maximal ideal of $\mathscr{A}$ such that $\operatorname{coker}(f)_{\mathfrak{M}} \neq 0$. Since f becomes an isomorphism after reduction to k, we have $\operatorname{coker}(f) \otimes_{\mathcal{O}_K} k = 0$, or equivalently $\operatorname{coker}(f) = \mathfrak{m} \operatorname{coker}(f)$. In particular, we find $\operatorname{coker}(f)_{\mathfrak{M}} = \mathfrak{m} \operatorname{coker}(f)_{\mathfrak{M}} \subseteq \mathfrak{M} \operatorname{coker}(f)_{\mathfrak{M}}$. Since $\operatorname{coker}(f)_{\mathfrak{M}}$ is finitely generated (by one element) over the local ring $\mathscr{A}_{\mathfrak{M}}$, we deduce $\operatorname{coker}(f)_{\mathfrak{M}} = 0$ by Nakayama's lemma, thereby obtaining the desired contradiction.

Let us now prove that f is injective. As in the previous paragraph, we find $\ker(f) = \mathfrak{m} \ker(f)$ by the fact that f becomes an isomorphism after reduction to k. Let us write $\mathscr{I} := (t_1, \cdots, t_d)$, and denote by $\widetilde{\mathscr{I}^j}$ the image of $\mathscr{I}^j$ under f. Then we have an exact sequence

$$0 \longrightarrow \ker(f)/\ker(f) \cap \mathscr{I}^j \longrightarrow \mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{I}^j \longrightarrow \mathscr{A}/\widetilde{\mathscr{I}^j} \longrightarrow 0.$$

Since $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{I}^j$ is noetherian, we can argue as in the preceding paragraph with the identity $\mathfrak{m}\left(\ker(f)/\ker(f) \cap \mathscr{I}^j\right) = \ker(f)/\ker(f) \cap \mathscr{I}^j$ to find $\ker(f) = \ker(f) \cap \mathscr{I}^j$. As $\cap_j \mathscr{I}^j = 0$, we deduce $\ker(f) = 0$ as desired.

Now, since f is an isomorphism as seen above, it yields a surjective map $\mathscr{A} \twoheadrightarrow \mathscr{A}^{\text{ét}}$ which splits the natural map $\mathscr{A}^{\text{ét}} \hookrightarrow \mathscr{A}$. We thus deduce the desired surjectivity of the map (3.3), thereby completing the proof. □

Corollary 3.2.6. *For every $x \in G(\mathcal{O}_L)$, we have $p^n x \in G^\circ(\mathcal{O}_L)$ for all sufficiently large n.*

PROOF. This is an immediate consequence of Proposition 3.2.3 and Proposition 3.2.5. □

Proposition 3.2.7. *Assume that L is algebraically closed. Then $G(\mathcal{O}_L)$ is p-divisible in the sense that the multiplication by p on $G(\mathcal{O}_L)$ is surjective.*

PROOF. By Proposition 3.2.5, it suffices to show the surjectivity of the multiplication by p on each $G^{\text{ét}}(\mathcal{O}_L)$ and $G^\circ(\mathcal{O}_L)$. The surjectivity on $G^{\text{ét}}(\mathcal{O}_L)$ is obvious by Corollary 2.1.6 and Proposition 3.2.3. Hence it remains to prove the surjectivity on $G^\circ(\mathcal{O}_L)$. Let us write $\mathscr{A}^\circ := \mathcal{O}_L[[t_1, \cdots, t_d]]$ where d is the dimension of G. Since the multiplication by p on $G^\circ(\mathcal{O}_L) \simeq \operatorname{Hom}_{\mathcal{O}_K-\text{cont}}(\mathscr{A}^\circ, \mathcal{O}_L)$ is induced by $[p]_{\mu(G)}$ on $\mathscr{A}^\circ$ as noted in Corollary 3.2.4, we deduce the desired surjectivity by the p-divisibility of $\mu(G)$. □

Remark. If we let $\mathscr{G}^\circ$ denote the formal group associated to G°, the surjectivity on $G^\circ(\mathcal{O}_L)$ also follows from the p-divisibility of $\mathscr{G}^\circ$ that we remarked after Theorem 2.2.16.

3.3. The logarithm for p-divisible groups

We retain the notations in the previous subsection. Our goal in this subsection is to construct and study the logarithm map for p-divisible groups over $\mathcal{O}_K$.

Definition 3.3.1. Let G be a p-divisible group over $\mathcal{O}_K$ of dimension d. Let us write $\mathscr{A}^\circ := \mathcal{O}_K[[t_1, \cdots, t_d]]$ and denote by $\mathscr{I}$ the augmentation ideal of $\mu(G)$.

(1) Given an $\mathcal{O}_K$-module M, we define the *tangent space of G with values in M* by

$$t_G(M) := \mathrm{Hom}_{\mathcal{O}_K\text{-mod}}(\mathscr{I}/\mathscr{I}^2, M),$$

and the *cotangent space of G with values in M* by

$$t_G^*(M) := \mathscr{I}/\mathscr{I}^2 \otimes_{\mathcal{O}_K} M.$$

(2) We define the *valuation filtration* of $G^\circ(\mathcal{O}_L)$ by setting

$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) := \{\, f \in G^\circ(\mathcal{O}_L) : \nu(f(x)) \geq \lambda \text{ for all } x \in \mathscr{I} \,\}$$

for all real number $\lambda > 0$, where we identify $G^\circ(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathscr{A}^\circ, \mathcal{O}_L)$ as described in Corollary 3.2.4.

Remark. We may identify t_G and t_G^* respectively with the tangent space and cotangent space of the formal group $\mathscr{G}_\mu$ induced by μ.

Lemma 3.3.2. *Let G be a p-divisible group over $\mathcal{O}_K$. For every $f \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$, we have $pf \in \mathrm{Fil}^\kappa G^\circ(\mathcal{O}_L)$ where $\kappa = \min(\lambda + 1, 2\lambda)$.*

Proof. Let $\mathscr{I}$ denote the augmentation ideal of $\mu(G)$. Lemma 2.2.14 yields $[p]_{\mu(G)}(x) = px + y$ for some $y \in \mathscr{I}^2$. We thus find

$$(pf)(x) = f([p]_{\mu(G)}(x)) = f(px + y) = pf(x) + f(y),$$

which implies $\nu((pf)(x)) \geq \min(\lambda + 1, 2\lambda)$ as desired. □

Lemma 3.3.3. *Let G be a p-divisible group over $\mathcal{O}_K$, and denote by $\mathscr{I}$ the augmentation ideal of $\mu(G)$. Let us choose arbitrary elements $f \in G(\mathcal{O}_L)$ and $x \in \mathscr{I}$. Then $\lim\limits_{n\to\infty} \dfrac{(p^n f)(x)}{p^n}$ exists in L, and equals zero if $x \in \mathscr{I}^2$.*

Proof. By Lemma 2.2.14 we may write $[p]_{\mu(G)}(x) = px+y$ for some $y \in \mathscr{I}^2$. In addition, by Corollary 3.2.6 we have $p^n f \in G^\circ(\mathcal{O}_L)$ for all sufficiently large n. Then an easy induction using Lemma 3.3.2 shows that there exists some constant c with $p^n f \in \mathrm{Fil}^{n+c} G^\circ(\mathcal{O}_L)$ for all sufficiently large n. Hence for all sufficiently large n we find

$$\frac{(p^{n+1} f)(x)}{p^{n+1}} - \frac{(p^n f)(x)}{p^n} = \frac{(p^n f)([p]_{\mu(G)}(x))}{p^{n+1}} - \frac{(p^n f)(x)}{p^n} = \frac{(p^n f)(y)}{p^{n+1}},$$

which in turn yields

$$\nu\left(\frac{(p^{n+1} f)(x)}{p^{n+1}} - \frac{(p^n f)(x)}{p^n}\right) \geq 2(n + c) - (n + 1) = n + (2c - 1).$$

Therefore the sequence $\left(\dfrac{(p^n f)(x)}{p^n}\right)$ converges in L for being Cauchy. Moreover, if $x \in \mathscr{I}^2$ the sequence converges to 0 as

$$\nu\left(\frac{(p^n f)(x)}{p^n}\right) \geq 2(n + c) - (n + 1) = n + (2c - 1)$$

for all sufficiently large n. □

Lemma 3.3.3 allows us to make the following definition.

Definition 3.3.4. Let G be a p-divisible group over $\mathcal{O}_K$, and let $\mathscr{I}$ denote the augmentation ideal of $\mu(G)$. We define the *logarithm* of G to be the map

$$\log_G : G(\mathcal{O}_L) \to t_G(L)$$

such that for every $f \in G(\mathcal{O}_L)$ and $x \in \mathscr{I}/\mathscr{I}^2$ we have

$$\log_G(f)(x) = \lim_{n\to\infty} \frac{(p^n f)(\tilde{x})}{p^n}$$

where $\tilde{x}$ is any lift of x to $\mathscr{I}$.

Remark. For curious readers, we describe an alternative construction of $\log_G$ using the theory of p-adic analytic groups. As remarked after Corollary 3.2.4, $G^\circ(\mathcal{O}_L)$ carries the structure of a p-adic analytic group over L. Moreover, we can identify its Lie algebra with $t_G(L)$. Hence we have a map $\log_{G^\circ} : G^\circ(\mathcal{O}_L) \to t_G(L)$ induced by the p-adic logarithm on the ambient analytic group. We then obtain $\log_G : G(\mathcal{O}_L) \to t_G(L)$ by setting $\log_G(f) := \dfrac{\widetilde{\log_{G^\circ}(p^n f)}}{p^n}$ for any $f \in G(\mathcal{O}_L)$ where n is chosen such that $p^n f$ belongs to $G^\circ(\mathcal{O}_L)$.

Example 3.3.5. Let us provide an explicit description of $\log_{\mu_{p^\infty}}$. As seen in Example 2.2.12, we have $\mu_{\widehat{\mathbb{G}}_m}[p^\infty] \cong \mu_{p^\infty}$. Corollary 3.2.4 then yields an identification

$$\mu_{p^\infty}(\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_L[[t]], \mathcal{O}_L) \cong \mathfrak{m}_L \cong 1 + \mathfrak{m}_L$$

where the last two isomorphisms are given by $f \mapsto f(t)$ and $x \mapsto 1 + x$. Note that this identification agrees with the identification obtained in Example 3.2.2. In addition, writing $\mathscr{I} := (t)$ for the augmentation ideal of $\mu_{\widehat{\mathbb{G}}_m}$ we find

$$t_{\mu_{p^\infty}}(L) = \mathrm{Hom}_{\mathcal{O}_K\text{-mod}}(\mathscr{I}/\mathscr{I}^2, L) \cong L.$$

We thus have a commutative diagram

$$\begin{array}{ccc} \mu_{p^\infty}(\mathcal{O}_L) & \xrightarrow{\log_{\mu_{p^\infty}}} & t_{\mu_{p^\infty}}(L) \\ {\scriptstyle f\mapsto 1+f(t)}\downarrow \wr & & \wr\downarrow {\scriptstyle g\mapsto g(t)} \\ 1+\mathfrak{m}_L & \longrightarrow & L \end{array} \tag{3.4}$$

Let us identify $\log_{\mu_{p^\infty}}$ with the bottom arrow. We also take an arbitrary element $1+x \in 1+\mathfrak{m}_L$. As each $f \in \mu_{p^\infty}(\mathcal{O}_L)$ satisfies

$$(p^n f)(t) = f\left([p^n]_{\widehat{\mathbb{G}}_m}(t)\right) = f\left((1+t)^{p^n} - 1\right) = (1 + f(t))^{p^n} - 1,$$

the diagram 3.4 yields an expression

$$\log_{\mu_{p^\infty}}(1+x) = \lim_{n\to\infty} \frac{(1+x)^{p^n} - 1}{p^n} = \lim_{n\to\infty} \sum_{i=1}^{p^n} \frac{1}{p^n}\binom{p^n}{i} x^i. \tag{3.5}$$

In addition, for each i and n we have

$$\frac{1}{p^n}\binom{p^n}{i} - \frac{(-1)^{i-1}}{i} = \frac{(p^n-1)\cdots(p^n-i+1) - (-1)^{i-1}(i-1)!}{i!}.$$

Since the numerator is divisible by p^n, we obtain an estimate

$$\nu\left(\frac{1}{p^n}\binom{p^n}{i}x^i - \frac{(-1)^{i-1}x^i}{i}\right) \geq n + i\nu(x) - \nu(i!) \geq n + i\nu(x) - \frac{i}{p-1}.$$

Hence we may write the expression (3.5) as

$$\log_{\mu_{p^\infty}}(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i,$$

which coincides with the p-adic logarithm.

Let us collect some basic properties of the logarithm for p-divisible groups.

Proposition 3.3.6. *Let G be a p-divisible group over $\mathcal{O}_K$. Denote by $\mathscr{I}$ the augmentation ideal of $\mu(G)$.*

(1) $\log_G$ *is a group homomorphism.*
(2) $\log_G$ *is a local isomorphism in the sense that for each real number $\lambda \geq 1$ it induces an isomorphism*

$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \xrightarrow{\sim} \left\{ \tau \in t_G(L) : \nu(\tau(x)) \geq \lambda \textit{ for all } x \in \mathscr{I}/\mathscr{I}^2 \right\}.$$

(3) The kernel of $\log_G$ *is the torsion subgroup $G(\mathcal{O}_L)_{\mathrm{tors}}$ of $G(\mathcal{O}_L)$.*
(4) $\log_G$ *induces an isomorphism $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq t_G(L)$.*

Proof. Let us write $\mathscr{A}^\circ := \mathcal{O}_K[[t_1, \cdots, t_d]]$ where d is the dimension of G. Take arbitrary elements $f, g \in G(\mathcal{O}_L)$ and $x \in \mathscr{I}$. Arguing as in Theorem 1.3.10, we find

$$\mu(G)(x) \in 1 \otimes x + x \otimes 1 + \mathscr{I} \widehat{\otimes}_{\mathscr{A}^\circ} \mathscr{I}.$$

Hence for all sufficiently large n we have

$$(p^n(f+g))(x) = (p^n f + p^n g)(x) = (p^n f \otimes p^n g) \circ \mu(x) = (p^n f)(x) + (p^n g)(x) + y$$

for some $y \in (p^n f)(\mathscr{I}) \cdot (p^n g)(\mathscr{I})$. Then a similar estimate as in Lemma 3.3.3 shows

$$\lim_{n \to \infty} \frac{(p^n(f+g))(x)}{p^n} = \lim_{n \to \infty} \frac{(p^n f)(x)}{p^n} + \lim_{n \to \infty} \frac{(p^n g)(x)}{p^n},$$

thereby implying that $\log_G$ is a homomorphism.

Let us now fix an arbitrary real number $\lambda \geq 1$ and write

$$\mathrm{Fil}^\lambda t_G(L) := \left\{ \tau \in t_G(L) : \nu(\tau(x)) \geq \lambda \text{ for all } x \in \mathscr{I}/\mathscr{I}^2 \right\}.$$

If $f \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$, Lemma 3.3.2 yields an estimate $\nu\left(\frac{(p^n f)(x)}{p^n}\right) \geq \lambda$ for all $x \in \mathscr{I}$ and $n > 0$, thereby implying $\log_G(f) \in \mathrm{Fil}^\lambda t_G(L)$. It is then straightforward to verify that $\log_G$ on $\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$ admits an inverse $\mathrm{Fil}^\lambda t_G(L) \to \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$ which sends each $\tau \in \mathrm{Fil}^\lambda t_G(L)$ to the unique $f \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$ with $f(t_i) = \tau(t_i)$. Therefore we deduce the statement (2).

Next we show $\ker(\log_G) = G(\mathcal{O}_L)_{\mathrm{tors}}$ as asserted in (3). We clearly have $G(\mathcal{O}_L)_{\mathrm{tors}} \subseteq \ker(\log_G)$ since $t_G(L)$ is torsion free for being a vector space over L. Hence we only need to establish the reverse inclusion $\ker(\log_G) \subseteq G(\mathcal{O}_L)$. Let f be an element in $\ker(\log_G)$. By (1) we have $p^n f \in \ker(\log_G)$ for all n. Moreover, Corollary 3.2.6 and Lemma 3.3.2 together yield $p^n f \in \mathrm{Fil}^1 G^\circ(\mathcal{O}_L)$ for all sufficiently large n. We then find $p^n f = 0$ for all sufficiently large n by (2), thereby deducing that f is a torsion element as desired.

Now (3) readily implies the injectivity of the map $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to t_G(L)$ induced by $\log_G$. We also deduce the surjectivity of the map from (2) by observing that every element $\tau \in t_G(L)$ satisfies $p^n \tau \in \mathrm{Fil}^1 t_G(L)$ for all sufficiently large n. □

3.4. Hodge-Tate decomposition for the Tate module

In this subsection, we derive the first main result for this chapter by exploiting our accumulated knowledge of finite flat group schemes and p-divisible groups.

Let us first present some easy but useful lemmas.

Lemma 3.4.1. *Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. For each v we have canonical isomorphisms*

$$G_v(\overline{K}) \cong G_v(\mathbb{C}_K) \cong G_v(\mathcal{O}_{\mathbb{C}_K}).$$

PROOF. Since $\mathbb{C}_K$ is algebraically closed as noted in Proposition 3.1.5, the first isomorphism follows from the fact that the generic fiber of G_v is étale by Corollary 1.3.11. The second isomorphism is a direct consequence of the valuative criterion. □

Lemma 3.4.2. *For every p-divisible group G over $\mathcal{O}_K$ we have*

$$G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} = G(\mathcal{O}_K) \qquad \text{and} \qquad t_G(\mathbb{C}_K)^{\Gamma_K} = t_G(K).$$

PROOF. By Theorem 3.1.12 we have $\mathbb{C}_K^{\Gamma_K} = K$ and $\mathcal{O}_{\mathbb{C}_K}^{\Gamma_K} = \mathcal{O}_K$. Hence the desired identifications immediately follow from Proposition 3.2.3 and Definition 3.3.1. □

Lemma 3.4.3. *Given a p-divisible group G over $\mathcal{O}_K$ we have*

$$\bigcap_{n=1}^{\infty} p^n G^\circ(\mathcal{O}_K) = 0.$$

PROOF. As the valuation on K is discrete, there exists a minimum positive valuation δ; indeed, we have $\delta = \nu(\pi)$ where π is a uniformizer of K. Then an easy induction using Lemma 3.3.2 yields $p^n G^\circ(\mathcal{O}_K) \subseteq \operatorname{Fil}^{n\delta} G^\circ(\mathcal{O}_K)$ for all $n \geq 1$. We thus deduce the desired assertion by observing $\displaystyle\bigcap_{n=1}^{\infty} \operatorname{Fil}^{n\delta} G^\circ(\mathcal{O}_K) = 0$. □

The main technical ingredient for this subsection is the interplay between the Tate modules and Cartier duality.

Definition 3.4.4. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$. We define the *Tate module* of G by

$$T_p(G) := T_p(G \times_{\mathcal{O}_K} K) = \varprojlim G_v(\overline{K}),$$

and the *Tate comodule* of G by

$$\Phi_p(G) := \varinjlim G_v(\overline{K}).$$

Remark. The Tate comodule $\Phi_p(G)$ is nothing other than $G(\overline{K})$, where G is regarded as a fpqc sheaf.

Example 3.4.5. We have $T_p(\mu_{p^\infty}) = \mathbb{Z}_p(1)$ as noted in Example 3.1.9. In addition, $\Phi(\mu_{p^\infty}) = \varinjlim \mu_{p^v}(\overline{K}) = \mu_{p^\infty}(\overline{K})$ is the group of p-power roots of unity in $\overline{K}$.

Proposition 3.4.6. *Given a p-divisible group G over $\mathcal{O}_K$, Cartier duality induces natural Γ_K-equivariant isomorphisms*

$$T_p(G) \cong \operatorname{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \qquad \text{and} \qquad \Phi_p(G) \cong \operatorname{Hom}_{\mathbb{Z}_p}\left(T_p(G^\vee), \mu_{p^\infty}(\overline{K})\right).$$

Proof. Note that every finite flat group scheme over $\overline{K}$ is étale by Corollary 1.3.11. For each v we have a natural identification

$$G_v(\overline{K}) \cong (G_v^\vee)^\vee(\overline{K}) = \mathrm{Hom}_{\overline{K}\text{-grp}}\left((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}\right) \cong \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K})) \tag{3.6}$$

by Theorem 1.2.3, Lemma 1.2.2, and Proposition 1.3.1. We then obtain a Γ_K-equivariant isomorphism

$$\begin{aligned} T_p(G) = \varprojlim G_v(\overline{K}) &\cong \varprojlim \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \varprojlim \mu_{p^v}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)). \end{aligned}$$

In addition, by writing (3.6) as $G_v(\overline{K}) \cong \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K}))$ we find another Γ_K-equivariant isomorphism

$$\begin{aligned} \Phi_p(G) = \varinjlim G_v(\overline{K}) &\cong \varinjlim \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K})) \\ &\cong \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \mu_{p^\infty}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mu_{p^\infty}(\overline{K})), \end{aligned}$$

thereby completing the proof. □

Proposition 3.4.7. *Let G be a p-divisible group over $\mathcal{O}_K$. We have an exact sequence*

$$0 \longrightarrow \Phi_p(G) \longrightarrow G(\mathcal{O}_{\mathbb{C}_K}) \xrightarrow{\log_G} \mathbb{C}_K \longrightarrow 0.$$

Proof. Since $G(\mathcal{O}_{\mathbb{C}_K})$ is p-divisible by Proposition 3.1.5 and Proposition 3.2.7, we obtain the surjectivity of $\log_G$ by Proposition 3.3.6. We then use Proposition 3.3.6, Proposition 3.2.3 and Lemma 3.4.1 to find

$$\ker(\log_G) = G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}} \cong \varinjlim_v \varprojlim_i G_v(\mathcal{O}_{\mathbb{C}_K}/\mathfrak{m}^i\mathcal{O}_{\mathbb{C}_K}) = \varinjlim_v G_v(\mathcal{O}_{\mathbb{C}_K}) \cong \varinjlim_v G_v(\overline{K}) = \Phi_p(G),$$

thereby completing the proof. □

Example 3.4.8. For $G = \mu_{p^\infty}$ Proposition 3.4.7 yields

$$0 \longrightarrow \mu_{p^\infty}(\overline{K}) \longrightarrow 1 + \mathfrak{m}_{\mathbb{C}_K} \xrightarrow{\log_{\mu_{p^\infty}}} \mathbb{C}_K \longrightarrow 0.$$

by Example 3.3.5 and Example 3.4.5.

Proposition 3.4.9. *Every p-divisible group G over $\mathcal{O}_K$ gives rise to a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \alpha & & \downarrow d\alpha & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G^\vee), \mu_{p^\infty}(\overline{K})\right) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}\right) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G^\vee), \mathbb{C}_K\right) & \longrightarrow & 0 \end{array}$$

where α and $d\alpha$ are Γ_K-equivariant and injective.

Proof. The top row is as described in Proposition 3.4.7. The bottom row is induced by the short exact sequence in Example 3.4.8, and is exact since $T_p(G^\vee)$ is free over $\mathbb{Z}_p$. The left vertical arrow is the natural Γ_K-equivariant isomorphism given by Proposition 3.4.6.

Let us now construct the maps α and $d\alpha$. As usual, we write $G = \varinjlim G_v$ where G_v is a finite flat $\mathcal{O}_K$-group scheme. Lemma 3.4.1 and Lemma 1.2.2 together yield

$$\begin{aligned} T_p(G^\vee) &= \varprojlim G_v^\vee(\overline{K}) \cong \varprojlim G_v^\vee(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varprojlim \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_K}\text{-grp}}\Big((G_v)_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^v})_{\mathcal{O}_{\mathbb{C}_K}}\Big) \\ &= \mathrm{Hom}_{p\text{-div grp}}\left(G \times_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_K}, (\mu_{p^\infty})_{\mathcal{O}_K}\right). \end{aligned} \tag{3.7}$$

We define the map $\alpha : G(\mathcal{O}_{\mathbb{C}_K}) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K})$ by setting

$$\alpha(g)(u) := u_{\mathcal{O}_{\mathbb{C}_K}}(g) \qquad \text{for each } g \in G(\mathcal{O}_{\mathbb{C}_K}) \text{ and } u \in T_p(G^\vee),$$

where $u_{\mathcal{O}_{\mathbb{C}_K}} : G(\mathcal{O}_{\mathbb{C}_K}) \to \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1 + \mathfrak{m}_{\mathbb{C}_K}$ is the map induced by u under the identification (3.7). We also define the map $d\alpha : t_G(\mathbb{C}_K) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ by setting

$$d\alpha(z)(u) := du_{\mathbb{C}_K}(z) \qquad \text{for each } z \in t_G(\mathbb{C}_K) \text{ and } u \in T_p(G^\vee),$$

where $du_{\mathbb{C}_K} : t_G(\mathbb{C}_K) \to t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K$ is the map induced by u under the identification (3.7).

The maps α and $d\alpha$ are evidently $\mathbb{Z}_p$-linear and Γ_K-equivariant by construction. The commutativity of the left square follows by observing that the left vertical arrow can be also defined as the restriction of α on $G(\mathcal{O}_{\mathbb{C}_K}) \cong \Phi_p(G)$. The commutativity of the right square amounts to the commutativity of the following diagram

$$\begin{array}{ccc} G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\ \log_G\ } & t_G(\mathbb{C}_K) \\ \big\downarrow & & \big\downarrow \\ \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) = 1 + \mathfrak{m}_{\mathbb{C}_K} & \xrightarrow{\ \log_{\mu_{p^\infty}}\ } & t_{\mu_{p^\infty}} = \mathbb{C}_K \end{array}$$

which is straightforward to verify by definition; indeed, the logarithm map yields a natural transformation between the functor of $\mathcal{O}_{\mathbb{C}_K}$-valued formal points and the functor of tangent space with values in K.

It remains to prove that α and $d\alpha$ are injective. By snake lemma we have $\mathbb{Z}_p$-linear isomorphisms

$$\ker(\alpha) \simeq \ker(d\alpha) \qquad \text{and} \qquad \mathrm{coker}(\alpha) \simeq \mathrm{coker}(d\alpha). \tag{3.8}$$

Hence it suffices to show that $d\alpha$ is injective.

As both $t_G(\mathbb{C}_K)$ and $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ are $\mathbb{Q}_p$-vector spaces, the $\mathbb{Z}_p$-linear map $d\alpha$ is indeed $\mathbb{Q}_p$-linear. Therefore both $\ker(d\alpha)$ and $\mathrm{coker}(d\alpha)$ are $\mathbb{Q}_p$-vector spaces. The isomorphisms (3.8) then tells us that both $\ker(\alpha)$ and $\mathrm{coker}(\alpha)$ are $\mathbb{Q}_p$-vector spaces as well.

We assert that α is injective on $G(\mathcal{O}_K)$. Suppose for contradiction that $\ker(\alpha)$ contains a nonzero element $g \in G(\mathcal{O}_K)$. As $\ker(\alpha)$ is torsion free for being a $\mathbb{Q}_p$-vector space, we may assume $g \in G^\circ(\mathcal{O}_K)$ by Corollary 3.2.6. Let us define the map

$$\alpha^\circ : G^\circ(\mathcal{O}_{\mathbb{C}_K}) \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p((G^\circ)^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K})$$

in the same way we define the map α. Since the natural map $T_p(G^\vee) \to T_p((G^\circ)^\vee)$ is surjective, we obtain a commutative diagram

$$\begin{array}{ccc} G^\circ(\mathcal{O}_{\mathbb{C}_K}) & \hookrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) \\ \big\downarrow{\scriptstyle \alpha^\circ} & & \big\downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p((G^\circ)^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) \end{array}$$

where both horizontal arrows are injective. In particular, we have $g \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$. Moreover, Lemma 3.4.2 yields $\ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K) = \ker(\alpha^\circ)^{\Gamma_K}$, which is a $\mathbb{Q}_p$-vector space since $\ker(\alpha^\circ)$ is a $\mathbb{Q}_p$-vector space by the same argument as in the preceding paragraph. Therefore for every $n \in \mathbb{Z}$ there exists an element $g_n \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$ with $g = p^n g_n$. However, this means $g = 0$ by Lemma 3.4.3, yielding the desired contradiction.

Next we show that $d\alpha$ is injective on $t_G(K)$. Since $\log_G(G(\mathcal{O}_K)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = t_G(K)$ by Proposition 3.3.6, it is enough to show the injectivity on $\log_G(G(\mathcal{O}_K))$. Choose an arbitrary element $h \in G(\mathcal{O}_K)$ such that $\log_G(h) \in \ker(d\alpha)$. We wish to show that $\log_G(h) = 0$. As the isomorphism $\ker(\alpha) \simeq \ker(d\alpha)$ in (3.8) is induced by $\log_G$, we can find $h' \in \ker(\alpha)$ with $\log_G(h) = \log_G(h')$. Then by Proposition 3.3.6 we have $h - h' \in \ker(\log_G) = G(\mathcal{O}_{\mathbb{C}_K})_{\text{tors}}$, which means that there exists some n with $p^n(h - h') = 0$, or equivalently $p^n h = p^n h'$. We thus find $p^n h \in \ker(\alpha) \cap G(\mathcal{O}_K)$, which implies $p^n h = 0$ by the injectivity of α on $G(\mathcal{O}_K)$. Hence we have $h \in G(\mathcal{O}_{\mathbb{C}_K})_{\text{tors}}$, thereby deducing $\log_G(h) = 0$ by Proposition 3.3.6.

As $t_G(K) = t_G(\mathbb{C}_K)^{\Gamma_K}$ by Lemma 3.4.2, we can factor $d\alpha$ as

$$d\alpha : t_G(\mathbb{C}_K) \cong t_G(K) \otimes_K \mathbb{C}_K \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K \to \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K).$$

The first arrow is injective by our discussion in the preceding paragraph. The second arrow is injective by Lemma 3.1.13 since we have a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), K) \otimes_K \mathbb{C}_K$$

due to the freeness of $T_p(G^\vee)$ over $\mathbb{Z}_p$. Hence we deduce the injectivity of $d\alpha$ as desired, thereby completing the proof. ☐

Theorem 3.4.10 (Tate [**Tat67**]). *Let G be a p-divisible group over $\mathcal{O}_K$. Define α and $d\alpha$ as in Proposition 3.4.9. Then their restrictions to the Γ_K-invariant elements yield bijective maps*

$$\begin{aligned} \alpha_K &: G(\mathcal{O}_K) \to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}), \\ d\alpha_K &: t_G(K) \to \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K). \end{aligned}$$

PROOF. By Proposition 3.4.9 we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{coker}(\alpha) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \log_G} & & \downarrow & & \downarrow{\scriptstyle \wr} & & \\ 0 & \longrightarrow & t_G(\mathbb{C}_K) & \xrightarrow{d\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \mathrm{coker}(d\alpha) & \longrightarrow & 0 \end{array}$$

where the bijectivity of the right vertical arrow follows from snake lemma as noted in (3.8). Taking Γ_K-invariants of the above diagram yields

$$\begin{array}{ccccccc} 0 & \longrightarrow & G(\mathcal{O}_K) & \xrightarrow{\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{coker}(\alpha)^{\Gamma_K} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & t_G(K) & \xrightarrow{d\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \mathrm{coker}(d\alpha)^{\Gamma_K} \end{array}$$

which implies the injectivity of α_K and $d\alpha_K$. Moreover, by the exactness of the middle terms we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{coker}(\alpha_K) & \hookrightarrow & \mathrm{coker}(\alpha)^{\Gamma_K} \\ \downarrow & & \downarrow \\ \mathrm{coker}(d\alpha_K) & \hookrightarrow & \mathrm{coker}(d\alpha)^{\Gamma_K} \end{array}$$

where the injectivity of the left vertical arrow follows from the injectivity of the other three arrows. Hence we only need to prove $\mathrm{coker}(d\alpha_K) = 0$, or equivalently the surjectivity of $d\alpha_K$.

Let h and d be the height and dimension of G, and let $d^\vee$ be the dimension of $G^\vee$. Note that the $\mathbb{C}_K$-vector spaces

$$V := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \qquad \text{and} \qquad W := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

are both h-dimensional. The injectivity of $d\alpha_K$ yields

$$\dim_K(W^{\Gamma_K}) \geq \dim_K(t_G(K)) = d \tag{3.9}$$

where equality holds if and only if $d\alpha_K$ is surjective. By switching the roles of G and $G^\vee$ we also find $\dim_K(V^{\Gamma_K}) \geq d^\vee$, thereby obtaining

$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \geq d + d^\vee = h \tag{3.10}$$

by Theorem 2.2.19.

By Proposition 3.4.6, we have a Γ_K-equivariant perfect pairing of $\mathbb{Z}_p$-modules

$$T_p(G) \times T_p(G^\vee) \to \mathbb{Z}_p(1).$$

The scalar extension to $\mathbb{C}_K$ of the dual pairing yields a Γ_K-equivariant $\mathbb{C}_K$-linear pairing

$$V \times W \to \mathbb{C}_K(-1), \tag{3.11}$$

which is perfect since both $T_p(G)$ and $T_p(G^\vee)$ are free over $\mathbb{Z}_p$. The image of $V^{\Gamma_K} \times W^{\Gamma_K}$ should lie in $\mathbb{C}_K(-1)^{\Gamma_K}$, which is zero by Theorem 3.1.12. This means that $V^{\Gamma_K} \otimes_K \mathbb{C}_K$ and $W^{\Gamma_K} \otimes_K \mathbb{C}_K$ are orthogonal under the perfect pairing (3.11), which further implies

$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \leq \dim_{\mathbb{C}_K}(V) = h.$$

We thus have equality in (3.10), which in turn implies equality in (3.9) and thereby yielding the desired surjectivity of $d\alpha_K$. □

Corollary 3.4.11. *For every p-divisible group G of dimension d over $\mathcal{O}_K$, we have an identity*

$$d = \dim_K(\mathrm{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G^\vee), \mathbb{C}_K)) = \dim_K(T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1))^{\Gamma_K}.$$

Proof. The first equality immediately follows from Theorem 3.4.10. The second equality follows by an identification

$$T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

where the isomorphisms are given by Proposition 3.4.6 and the freeness of $T_p(G^\vee)$ over $\mathbb{Z}_p$. □

We are finally ready to prove the first main result for this chapter.

Theorem 3.4.12 (Tate [**Tat67**]). *Let G be a p-divisible group over $\mathcal{O}_K$. There is a canonical isomorphism of $\mathbb{C}_K[\Gamma_K]$-modules*

$$\mathrm{Hom}(T_p(G), \mathbb{C}_K) \cong t_{G^\vee}(\mathbb{C}_K) \oplus t_G^*(\mathbb{C}_K)(-1).$$

Proof. Theorem 3.4.10 yields natural isomorphisms

$$t_G(\mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K,$$
$$t_{G^\vee}(\mathbb{C}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K.$$

Moreover, the proof of Theorem 3.4.10 shows that $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal under the perfect pairing

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \times \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \to \mathbb{C}_K(-1)$$

as constructed in (3.11), with equality

$$\dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K)) + \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K)) = \dim_{C_K}(\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)).$$

This means that $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal complements with respect to the above pairing, thereby yielding an exact sequence

$$0 \longrightarrow t_{G^\vee}(\mathbb{C}_K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \longrightarrow t_G^*(\mathbb{C}_K)(-1) \longrightarrow 0 \tag{3.12}$$

where for the last term we use the identification $\mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1)) \cong t_G^*(\mathbb{C}_K)(-1)$ that follows by observing that $t_G^*(\mathbb{C}_K)$ is the $\mathbb{C}_K$-dual $t_G(\mathbb{C}_K)$. Writing $d := \dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K))$ and $d^\vee := \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K))$ we find

$$\begin{aligned}\mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(t_G^*(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) &\simeq \mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \\ &\simeq H^1(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee} = 0\end{aligned}$$

by Theorem 3.1.12, thereby deducing that the exact sequence (3.12) splits. Moreover, such a splitting is unique since we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(t_G^*(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) &\simeq \mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \\ &\simeq H^0(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee} = 0\end{aligned}$$

by Theorem 3.1.12. Hence we obtain the desired assertion. □

Definition 3.4.13. Given a p-divisible group G over $\mathcal{O}_K$, we refer to the isomorphism in Theorem 3.4.12 as the *Hodge-Tate decomposition* for G.

Corollary 3.4.14. *For every p-divisible group G over $\mathcal{O}_K$, the rational Tate-module*

$$V_p(G) := V_p(G \times_{\mathcal{O}_K} K) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a Hodge-Tate p-adic representation of Γ_K.

Proof. As the $\mathbb{C}_K$-duals of $t_{G^\vee}(\mathbb{C}_K)$ and $t_G^*(\mathbb{C}_K)$ are respectively given by $t_{G^\vee}^*(\mathbb{C}_K)$ and $t_G(\mathbb{C}_K)$, Theorem 3.4.12 yields a decomposition

$$V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong t_{G^\vee}^*(\mathbb{C}_K) \oplus t_G(\mathbb{C}_K)(1).$$

Then for each n we find

$$\big(V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\big)^{\Gamma_K} \cong \begin{cases} (t_{G^\vee}^*(\mathbb{C}_K) & \text{if } n = 0, \\ t_G(\mathbb{C}_K) & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

by Theorem 3.1.12. The assertion is now obvious by Definition 3.1.6. □

Let us conclude this subsection with an geometric application of Theorem 3.4.12.

Proposition 3.4.15. *Let A be an abelian variety over K with good reduction. Then we have a canonical Γ_K-equivariant isomorphism*

$$H^n_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(A, \Omega^j_{A/K}) \otimes_K \mathbb{C}_K(-j).$$

Proof. Let $A^\vee$ denote the dual abelian variety of A. Since A has good reduction, there exists an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ with $\mathcal{A}_K \cong A$. Then we have $T_p(\mathcal{A}[p^\infty]) = T_p(A[p^\infty])$ by definition, and $\mathcal{A}^\vee[p^\infty] \cong \mathcal{A}[p^\infty]^\vee$ as noted in Example 2.1.9. In addition, we have the following standard facts:

(1) There is a canonical isomorphism

$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A[p^\infty]), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

(2) The formal completion of $\mathcal{A}$ along the unit section gives rise to the formal group law $\mu(\mathcal{A}[p^\infty])$.

(3) There are canonical isomorphisms

$$H^0(A, \Omega^1_{A/K}) \cong t_e^*(A) \qquad \text{and} \qquad H^1(A, \mathcal{O}_A) \cong t_e(A^\vee)$$

where $t_e^*(A)$ and $t_e(A)$ respectively denote the cotangent space of A and tangent space of $A^\vee$ (at the unit section).

(4) We have identifications

$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \bigwedge^n H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p),$$
$$H^i(A, \Omega^j_{A/K}) \cong \bigwedge^i H^1(A, \mathcal{O}_A) \otimes \bigwedge^j H^0(A, \Omega^1_{A/K}).$$

The statements (2) and (3) together yield identifications

$$H^0(A, \Omega^1_{A/K}) \cong t^*_{\mathcal{A}[p^\infty]}(K) \qquad \text{and} \qquad H^1(A, \mathcal{O}_A) \cong t_{\mathcal{A}^\vee[p^\infty]}(K).$$

Hence Theorem 3.4.12 yields a canonical Γ_K-equivariant isomorphism

$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong (H^1(A, \mathcal{O}_A) \otimes_K \mathbb{C}_K) \oplus (H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_K(-1)).$$

We then obtain the desired isomorphism by (4). □

Remark. Proposition 3.4.15 is a special case of the general Hodge-Tate decomposition theorem that we introduced in Chapter I, Theorem 1.2.1. The original proof by Faltings in [**Fal88**] relies on the language of almost mathematics. Recently, inspired by the work of Faltings, Scholze [**Sch13**] extended the Hodge-Tate decomposition theorem to rigid analytic varieties using his theory of perfectoid spaces. A good exposition of Scholze's work can be found in Bhatt's notes [**Bha**].

Corollary 3.4.16. *For every abelian variety A over K with good reduction, the étale cohomology $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is a Hodge-Tate p-adic representation of Γ_K.*

PROOF. For each $j \in \mathbb{Z}$ we find

$$\left(H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(j)\right)^{\Gamma_K} \cong \begin{cases} H^{n-j}(A, \Omega^j_{A/K}) & \text{if } 0 \leq j \leq n, \\ 0 & \text{otherwise} \end{cases}$$

by Proposition 3.4.15 and Theorem 3.1.12. Hence we deduce the desired assertion by Definition 3.1.6. □

Remark. Corollary 3.4.16 readily extends to an arbitrary proper smooth variety X over K, as for each $j \in \mathbb{Z}$ the general Hodge-Tate decomposition theorem and Theorem 3.1.12 together yield an identification

$$\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(j)\right)^{\Gamma_K} \cong \begin{cases} H^{n-j}(X, \Omega^j_{X/K}) & \text{if } 0 \leq j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the above identification shows that $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ recovers the Hodge number (and Hodge cohomology) of X. This is a p-adic analogue of the fact from the classical Hodge theory that the Hodge numbers are topological invariants of a smooth proper variety over $\mathbb{C}$.

3.5. Generic fibers of p-divisible groups

The main focus of this subsection is to prove the second main result for this chapter, which says that the generic fiber functor on the category of p-divisible groups over $\mathcal{O}_K$ is fully faithful.

We assume the following technical result without proof.

Proposition 3.5.1. *Let $G = \varinjlim G_v$ be a p-divisible group of height h and dimension d over $\mathcal{O}_K$. Let us $G_v = \operatorname{Spec}(A_v)$ where A_v is a finite free $\mathcal{O}_K$-algebra. Then the discriminant ideal of A_v over $\mathcal{O}_K$ is generated by $p^{dvp^{hv}}$.*

Remark. For curious readers, we briefly sketch the proof of Proposition 3.5.1. Let $\operatorname{disc}(A_v)$ denote the discriminant ideal of A_v over $\mathcal{O}_K$. By Proposition 2.1.5 we have a short exact sequence

$$\underline{0} \longrightarrow G_1 \longrightarrow G_{v+1} \longrightarrow G_v \longrightarrow \underline{0}.$$

From this we can deduce a relation $\operatorname{disc}(A_{v+1}) = \operatorname{disc}(A_v)^{p^v} \operatorname{disc}(A_1)^{p^{hv}}$, thereby reducing our proof to the case $v = 1$. Moreover, if we write $G_1^\circ = \operatorname{Spec}(A_1^\circ)$ we can find $\operatorname{disc}(A_1^\circ) = \operatorname{disc}(A_1)$ from the connected-étale sequence of G_1. Hence it suffices to consider the case where G is connected. Let us write $\mathscr{A} := \mathcal{O}_K[[t_1, \cdots, t_d]]$ and $\mathscr{I} = (t_1, \cdots, t_d)$. Then we have $A_1 \simeq \mathcal{O}_K \otimes_{\mathscr{A}, [p]_{\mu(G)}} \mathscr{A}$ as shown in the proof of Proposition 2.2.10. Therefore we can compute $\operatorname{disc}(A_1)$ by the discriminant ideal of $\mathscr{A}$ over $[p]_{\mu(G)}(\mathscr{A})$. However, computing the discriminant ideal of $\mathscr{A}$ over $[p]_{\mu(G)}(\mathscr{A})$ turns out to be extremely technical; the best reference that we can provide here is Haines' notes [**Hai**, §2.3]

Our main strategy is to work on the level of Tate modules. The key ingredient is the fact that, for p-divisible groups over $\mathcal{O}_K$, the maps on the generic fibers are completely determined by the maps on the Tate modules by Proposition 2.1.14. Here we present two consequences of this fact as preparation for the proof of the main result.

Lemma 3.5.2. *Let $f : G \to H$ be a homomorphism of p-divisible groups over $\mathcal{O}_K$. If the restriction of f on the generic fibers is an isomorphism, then f is an isomorphism.*

Proof. Let us write $G = \varinjlim G_v$ and $H = \varinjlim H_v$ where $G_v = \operatorname{Spec}(A_v)$ and $H_v = \operatorname{Spec}(B_v)$ are finite flat group schemes over $\mathcal{O}_K$. Let $\alpha_v : B_v \to A_v$ be the map of $\mathcal{O}_K$-algebras induced by f. We wish to show that α_v is an isomorphism. Since $\alpha_v \otimes 1 : B_v \otimes_{\mathcal{O}_K} K \to A_v \otimes_{\mathcal{O}_K} K$ is an isomorphism, α_v must be injective by the freeness of B_v over $\mathcal{O}_K$. Hence it suffices to show that A_v and B_v have the same discriminant ideal over $\mathcal{O}_K$.

As the generic fibers $G \times_{\mathcal{O}_K} K$ and $H \times_{\mathcal{O}_K} K$ are isomorphic, we have $T_p(G) \simeq T_p(H)$. In particular, by Corollary 3.4.11 we find that G and H have the same height and dimension. The desired assertion now follows from Proposition 3.5.1. □

Remark. As the proof of Lemma 3.5.2 shows, Corollary 3.4.11 and Proposition 3.5.1 are the main technical inputs for our main result in this subsection. They reflect Tate's key insight that the dimension of a p-divisible group should be encoded in the Tate module. Theorem 3.4.12 was indeed discovered as a byproduct in an attempt to verify his insight.

Proposition 3.5.3. *Let G be a p-divisible group over $\mathcal{O}_K$, and let M a $\mathbb{Z}_p$-direct summand of $T_p(G)$ which is stable under the action of Γ_K. There exists a p-divisible group H over $\mathcal{O}_K$ with a homomorphism $\iota : H \to G$ which induces an isomorphism $T_p(H) \simeq M$.*

Proof. As usual, let us write $G = \varinjlim G_v$ where G_v is a finite flat group scheme over $\mathcal{O}_K$. By Proposition 2.1.14, the submodule M of $T_p(G)$ gives rise to a p-divisible group

$\widetilde{H} = \varinjlim \widetilde{H}_v$ over K with a homomorphism $\widetilde{H} \to G \times_{\mathcal{O}_K} K$ which induces a closed embedding $\widetilde{\iota}_v : \widetilde{H}_v \hookrightarrow G_v \times_{\mathcal{O}_K} K$ at each finite level. Let h be the height of $\widetilde{H}$, and let $\underline{H_v}$ denote the scheme theoretic closure of $\widetilde{H}_v$ in G_v. We then quickly verify that $\underline{H}_v$ is a finite flat group scheme of order p^{vh}. Moreover, the closed embedding $\widetilde{H}_v \hookrightarrow \widetilde{H}_{v+1}$ extends to a closed embedding $\underline{H}_v \hookrightarrow \underline{H}_{v+1}$.

Let us now consider the quotient $\underline{H}_{v+1}/\underline{H}_v$. Observe that $[p]$ factors through the unit section on the generic fiber $\widetilde{H}_{v+1}/\widetilde{H}_v \simeq \widetilde{H}_1 = \widetilde{H}[p]$. Passing to the scheme theoretic closure, we find that $[p]$ also factors through the unit section on $\underline{H}_{v+1}/\underline{H}_v$. Therefore $[p]_{\underline{H}_{v+2}}$ induces a homomorphism

$$\delta_v : \underline{H}_{v+2}/\underline{H}_{v+1} \to \underline{H}_{v+1}/\underline{H}_v$$

which yields an isomorphism on the generic fibers. Let us write $\underline{H}_{v+1}/\underline{H}_v = \mathrm{Spec}\,(B_v)$ where B_v is a finite free $\mathcal{O}_K$-algebra. The map $B_v \to B_{v+1}$ induced by δ_v is injective, as it becomes isomorphism upon tensoring with K. Hence the B_v's form an increasing sequence of $\mathcal{O}_K$-orders in the K-algebra $B_1 \otimes_{\mathcal{O}_K} K$. In addition, since $B_1 \otimes_{\mathcal{O}_K} K$ is finite étale by Corollary 1.3.11, we adapt the argument of [**AM94**, Proposition 5.17] to deduce that the integral closure of $\mathcal{O}_K$ in $B_1 \otimes_{\mathcal{O}_K} K$ is noetherian. Therefore there exists some v_0 such that $B_v \simeq B_{v+1}$ for all $v \geq v_0$, or equivalently δ_v is an isomorphism for all $v \geq v_0$.

Let us set $H_v := \underline{H}_{v_0+v}/\underline{H}_{v_0}$. We have a closed embedding $H_v \hookrightarrow H_{v+1}$ induced by the closed embedding $\underline{H}_{v_0+v} \hookrightarrow \underline{H}_{v_0+v+1}$. We assert that $H := \varinjlim H_v$ is a p-divisible group over $\mathcal{O}_K$. By construction, H_v is a finite flat $\mathcal{O}_K$-group scheme of order p^{vh}. Moreover, we have a commutative diagram

$$\begin{array}{ccc} H_{v+1} = \underline{H}_{v_0+v+1}/\underline{H}_{v_0} & \xrightarrow{[p^v]} & \underline{H}_{v_0+v+1}/\underline{H}_{v_0} = H_{v+1} \\ \downarrow\!\!\!\downarrow & & \uparrow \\ \underline{H}_{v_0+v+1}/\underline{H}_{v_0+v} & \xrightarrow{\sim} & \underline{H}_{v_0+1}/\underline{H}_{v_0} = H_1 \end{array}$$

where the bottom arrow is given by $\delta_{v_0} \circ \cdots \circ \delta_{v_0+v}$. We then find that the kernel of $[p^v]$ on H_{v+1} is equal to the kernel of the left vertical arrow, thereby deducing $H_{v+1}[p^v] = \underline{H}_{v_0+v}/\underline{H}_{v_0} = H_v$.

We now define a homomorphism $\iota_v : H_v \to G_v$ by the composition

$$H_v = \underline{H}_{v_0+v}/\underline{H}_{v_0} \xrightarrow{[p^{v_0}]} \underline{H}_v \hookrightarrow G_v.$$

It is straightforward to check that the maps ι_v give rise to a homomorphism $\iota : H \to G$. Moreover, on the generic fibers it induces a map

$$\widetilde{H}_{v_0+v}/\widetilde{H}_{v_0} \xrightarrow{[p^{v_0}]} \widetilde{H}_v \hookrightarrow G_v \times_{\mathcal{O}_K} K$$

where the first arrow is an isomorphism by the p-divisibility of $\widetilde{H}$. Hence we find that ι induces an isomorphism $T_p(H) \simeq T_p(\widetilde{H}) \simeq M$, thereby completing the proof. □

Let us now prove the second main result of this chapter.

Theorem 3.5.4 (Tate [**Tat67**]). *For arbitrary p-divisible groups G and H over $\mathcal{O}_K$, the natural map*

$$\mathrm{Hom}(G, H) \to \mathrm{Hom}(G \times_{\mathcal{O}_K} K, H \times_{\mathcal{O}_K} K)$$

is bijective.

Proof. Let us write $G = \varinjlim G_v$ and $H = \varinjlim H_v$ where $G_v = \operatorname{Spec}(A_v)$ and $H_v = \operatorname{Spec}(B_v)$ are finite flat group schemes over $\mathcal{O}_K$. Consider an arbitrary homomorphism $\widetilde{f} : G \times_{\mathcal{O}_K} K \to H \times_{\mathcal{O}_K} K$. We wish to show that $\widetilde{f}$ uniquely extends to a homomorphism $f : G \to H$.

Let $\widetilde{\alpha}_v : B_v \otimes_{\mathcal{O}_K} K \to A_v \otimes_{\mathcal{O}_K} K$ be the map of K-algebras induced by $\widetilde{f}$. As B_v is free over $\mathcal{O}_K$, there exists at most one $\mathcal{O}_K$-algebra homomorphism $\alpha_v : B_v \to A_v$ such that $\alpha_v \otimes 1 = \widetilde{\alpha}_v$. Hence we deduce that there exists at most one homomorphism $f : G \to H$ that extends $\widetilde{f}$.

It remains to construct an extension $f : G \to H$ of $\widetilde{f}$. Recall that $T_p(G \times_{\mathcal{O}_K} K) = T_p(G)$ and $T_p(H \times_{\mathcal{O}_K} K) = T_p(H)$ by definition. Let $\tau : T_p(G) \to T_p(H)$ be the map on the Tate modules induced by $\widetilde{f}$. Denote by M the graph of τ in $T_p(G) \oplus T_p(H)$. Clearly M is a $\mathbb{Z}_p[\Gamma_K]$-submodule of $T_p(G) \oplus T_p(H)$. Moreover, the quotient $(T_p(G) \oplus T_p(H))/M$ is torsion-free as there is an injective $\mathbb{Z}_p$-linear map

$$(T_p(G) \oplus T_p(H))/M \hookrightarrow T_p(H)$$

defined by $(x, y) \mapsto y - \tau(x)$. Since $\mathbb{Z}_p$ is a principal ideal domain, we find that $(T_p(G) \oplus T_p(H))/M$ is free over $\mathbb{Z}_p$, thereby deducing that the exact sequence

$$0 \longrightarrow M \longrightarrow T_p(G) \oplus T_p(H) \longrightarrow (T_p(G) \oplus T_p(H))/M \longrightarrow 0$$

splits. This means that M is a $\mathbb{Z}_p$-direct summand of $T_p(G) \oplus T_p(H) \cong T_p(G \times_{\mathcal{O}_K} H)$. Hence Proposition 3.5.3 yields a p-divisible group G' over $\mathcal{O}_K$ with a homomorphism $\iota : G' \to G \times_{\mathcal{O}_K} H$ which induces an isomorphism $T_p(G') \simeq M$. Let us now consider the projection maps $\pi_1 : G \times_{\mathcal{O}_K} H \twoheadrightarrow G$ and $\pi_2 : G \times_{\mathcal{O}_K} H \twoheadrightarrow H$. The map $\pi_1 \circ \iota$ induces an isomorphism $T_p(G') \simeq T_p(G)$ by construction, and thus induces an isomorphism on the generic fibers by Proposition 2.1.14. Hence Lemma 3.5.2 implies that $\pi_1 \circ \iota$ is an isomorphism. We then find that $f := \pi_2 \circ \iota \circ (\pi_1 \circ \iota)^{-1}$ induces the map τ on the Tate modules by construction, and thereby extends $\widetilde{f}$ by Proposition 2.1.14. □

Remark. As a related fact, the special fiber functor on the category of p-divisible groups over $\mathcal{O}_K$ is faithful. In other words, for arbitrary p-divisible groups G and H over $\mathcal{O}_K$, the natural map

$$\operatorname{Hom}(G, H) \to \operatorname{Hom}(G \times_{\mathcal{O}_K} k, H \times_{\mathcal{O}_K} k)$$

is injective. A complete proof of this fact can be found in [**CCO14**, Proposition 1.4.2.3].

It is also worthwhile to mention that Theorem 3.5.4 remains true if the base ring $\mathcal{O}_K$ is replaced by any ring R that satisfies the following properties:

(i) R is integrally closed and noetherian,
(ii) R is an integral domain whose fraction field has characteristic 0.

In fact, it is not hard to deduce the general case from Theorem 3.5.4 by algebraic Hartog's Lemma.

Corollary 3.5.5. *For arbitrary p-divisible groups G and H over $\mathcal{O}_K$, the natural map*

$$\operatorname{Hom}(G, H) \to \operatorname{Hom}_{\mathbb{Z}_p[\Gamma_K]}(T_p(G), T_p(H))$$

is bijective.

Proof. This is an immediate consequence of Proposition 2.1.14 and Theorem 3.5.4. □

We conclude this section by stating a fundamental theorem which provides a classification of p-divisible groups over $\mathcal{O}_K$ when K is unramified over $\mathbb{Q}_p$. We write $W(k)$ for the ring of Witt vectors over k.

Definition 3.5.6. A *Honda system* over $W(k)$ is a Dieudonné module M over k together with a $W(k)$-submodule L such that φ_M induces an isomorphism $L/pL \simeq M/\varphi_M(M)$.

Theorem 3.5.7 (Fontaine [**Fon77**]). *If $p > 2$, there exists an anti-equivalence of categories*

$$\{ \text{ } p\text{-divisible groups over } W(k) \text{ } \} \xrightarrow{\sim} \{ \text{ Honda systems over } W(k) \text{ } \}$$

such that for every p-divisible group G over $W(k)$ with the mod p reduction $\overline{G} := G \times_{W(k)} k$, the Dieudonné module of the associated Honda system coincides with $\mathbb{D}(\overline{G})$.

Remark. Let A be an abelian variety over K with good reduction. This means that there exists an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ with $\mathcal{A}_K \cong A$. As noted in the proof of Proposition 3.4.15, we have canonical identifications

$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \bigwedge^n H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p),$$
$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(\mathcal{A}[p^\infty]), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Hence Corollary 3.5.5 implies that the Γ_K-action on $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is determined by $\mathcal{A}[p^\infty]$.

Let us now assume that K is an unramified extension of $\mathbb{Q}_p$ with $p > 2$. Then we can identify $\mathcal{O}_K$ with the ring of Witt vectors over k. Therefore we deduce from Theorem 3.5.7 that $\mathcal{A}[p^\infty]$ is determined by the Dieudonné module $\mathbb{D}(\mathcal{A}_k[p^\infty])$ over k equipped with some filtration. This implies that the study of the Γ_K-action on $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is equivalent to the study of the Dieudonné module $\mathbb{D}(\mathcal{A}_k[p^\infty])$ over k equipped with some filtration.

Note that our discussion in the preceding paragraph recovers the toy example that we described in §1.1 of Chapter I as a special case. Furthermore, it turns out that the canonical isomorphism

$$H^1_{\text{cris}}(\mathcal{A}_k/\mathcal{O}_K) \cong \mathbb{D}(\mathcal{A}_k[p^\infty])$$

that we remarked after Example 2.3.20 is compatible with the filtrations on both sides. Hence we can rephrase the conclusion of the preceding paragraph as an equivalence between the study of the Γ_K-action on $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ and the study of $H^1_{\text{cris}}(\mathcal{A}_k/\mathcal{O}_K)$.

The discovery of this equivalence is what motivated the "mysterious functor" conjecture and ultimately led to the crystalline comparison theorem as stated in Conjecture 1.2.3 and Theorem 1.2.4 of Chapter I. In fact, in light of the canonical isomorphism

$$H^n_{\text{cris}}(\mathcal{A}_k/\mathcal{O}_K) \cong \bigwedge^n H^1_{\text{cris}}(\mathcal{A}_k/\mathcal{O}_K),$$

the equivalence that we discussed above can be realized as a special case of the crystalline comparison theorem.

CHAPTER III

Period rings and functors

1. Fontaine's formalism on period rings

In this section, we discuss some general formalism for p-adic period rings and period functors, as originally developed by Fontaine in [**Fon94**]. Our primary reference for this section Brinon and Conrad's notes [**BC**, §5].

1.1. Basic definitions and examples

Throughout this chapter, we let K be a p-adic field with the absolute Galois group Γ_K, the inertia group I_K, and the residue field k. We also denote by χ the p-adic cyclotomic character of K as defined in Chapter II, Example 3.1.9.

Definition 1.1.1. Let B be a $\mathbb{Q}_p$-algebra with an action of Γ_K. We denote by C the fraction field of B, endowed with a natural action of Γ_K which extends the action on B. We say that B is $(\mathbb{Q}_p, \Gamma_K)$-*regular* if it satisfies the following conditions:

(i) We have an identity $B^{\Gamma_K} = C^{\Gamma_K}$.

(ii) An element $b \in B$ is a unit if the set

$$\mathbb{Q}_p \cdot b := \{ \, c \cdot b : c \in \mathbb{Q}_p \, \}$$

is stable under the action of Γ_K.

Remark. For any field F and any group G, we can similarly define the notion of (F, G)-regular rings. Then the formalism that we develop in this section readily extends to (F, G)-regular rings. In particular, the topologies on $\mathbb{Q}_p$ and Γ_K do not play any role in our formalism.

Example 1.1.2. Every field extension of $\mathbb{Q}_p$ with an action of Γ_K is $(\mathbb{Q}_p, \Gamma_K)$-regular, as easily seen by Definition 1.1.1.

Definition 1.1.3. Let B be a $(\mathbb{Q}_p, \Gamma_K)$-regular ring. Let us write $E := B^{\Gamma_K}$, and denote by Vec_E the category of finite dimensional vector spaces over E.

(1) We define the functor $D_B : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K) \longrightarrow \mathrm{Vec}_E$ by

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \qquad \text{for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K).$$

(2) We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if it satisfies

$$\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

(3) We write $\mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ for the category of B-admissible p-adic Γ_K-representations.

Remark. Let us briefly describe a cohomological interpretation of the notion of B-admissibility. For any topological ring R with an action of Γ_K, there is a natural bijection between the pointed set $H^1(\Gamma_K, \mathrm{GL}_d(R))$ and the set of isomorphism classes of continuous semilinear Γ_K-representation over R of rank d. Hence every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ corresponds to a class $[V] \in H^1(\Gamma_K, \mathrm{GL}_d(\mathbb{Q}_p))$, which in turn gives rise to a class $[V]_B \in H^1(\Gamma_K, \mathrm{GL}_d(B))$. It turns out that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-admissible if and only if $[V]_B$ is trivial.

Example 1.1.4. We record some simple (but not necessarily easy) examples of admissible representations.

(1) For every $(\mathbb{Q}_p, \Gamma_K)$-regular ring B we have $\mathbb{Q}_p \in \mathrm{Rep}^B_{\mathbb{Q}_p}(\Gamma_K)$ with $D_B(\mathbb{Q}_p) = B^{\Gamma_K}$.
(2) Every p-adic representation is $\mathbb{Q}_p$-admissible, as $D_{\mathbb{Q}_p}$ is the identity functor.
(3) Essentially by Hilbert's Theorem 90, a p-adic representation V of Γ_K is $\overline{K}$-admissible if and only if V is *potentially trivial* in the sense that the action of Γ_K on V factors through a finite quotient.
(4) By a hard result of Sen, a p-adic representation V of Γ_K is $\mathbb{C}_K$-admissible if and only if V is *potentially unramified* in the sense that the action of I_K on V factors through a finite quotient.

We now describe how Hodge-Tate representations fit into the formalism that we have developed so far.

Definition 1.1.5. Let $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ be a character. For every $\mathbb{Q}_p[\Gamma_K]$-module M, we define its *twist by* η to be the $\mathbb{Q}_p[\Gamma_K]$-module

$$M(\eta) := M \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)$$

where $\mathbb{Q}_p(\eta)$ denotes the Γ_K-representation on $\mathbb{Q}_p$ given by η.

Example 1.1.6. Given a $\mathbb{Q}_p[\Gamma_K]$-module M, we have an identification $M(n) \cong M(\chi^n)$ for every $n \in \mathbb{Z}$ by Lemma 3.1.11 in Chapter II.

Lemma 1.1.7. *The group $\chi(I_K)$ is infinite.*

Proof. By definition χ encodes the action of Γ_K on $\mu_{p^\infty}(\overline{K})$. In particular, we have $\ker(\chi) = \mathrm{Gal}(K(\mu_{p^\infty}(\overline{K}))/K)$. Hence it suffices to show that $K(\mu_{p^\infty}(\overline{K}))$ is infinitely ramified over K.

Let e_n be the ramification degree of $K(\mu_{p^n}(\overline{K}))$ over K, and let e be the ramification degree of K over $\mathbb{Q}_p$. Then $e_n \cdot e$ is greater than or equal to the ramification degree of $\mathbb{Q}_p(\mu_{p^{n-1}}(\overline{K}))$ over $\mathbb{Q}_p$, which is equal to $p^{n-1}(p-1)$. We thus find that e_n grows arbitrarily large as n goes to ∞, thereby deducing the desired assertion. □

Theorem 1.1.8 (Tate [**Tat67**])**.** *Let $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^\times$ be a continuous character. Then for $i = 0, 1$ we have canonical isomorphisms*

$$H^i(\Gamma_K, \mathbb{C}_K(\eta)) \cong \begin{cases} K & \text{if } \eta(I_K) \text{ is finite,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark. Theorem 1.1.8 recovers the essential part of the Tate-Sen theorem as stated in Chapter II, Theorem 3.1.12; indeed, if we take $\eta = \chi^n$ for some $n \in \mathbb{Z}$, then Theorem 1.1.8 yields canonical isomorphisms

$$H^0(\Gamma_K, \mathbb{C}_K(n)) \cong H^1(\Gamma_K, \mathbb{C}_K(n)) \cong \begin{cases} K & \text{for } n = 0, \\ 0 & \text{for} n \neq 0, \end{cases}$$

by Example 1.1.6 and Lemma 1.1.7. Moreover, for $i = 0$ Theorem 1.1.8 says that $\mathbb{Q}_p(\eta)$ is $\mathbb{C}_K$-admissible if and only if it is potentially unramified, as we have already mentioned in Example 1.1.4.

Definition 1.1.9. We define the *Hodge-Tate period ring* by

$$B_{\mathrm{HT}} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_K(n).$$

Proposition 1.1.10. *The Hodge-Tate period ring B_{HT} is $(\mathbb{Q}_p, \Gamma_K)$-regular.*

PROOF. Let us first check the condition (i) in Definition 1.1.1. Let C_{HT} denote the fraction field of B_{HT}. Since we have $B_{\mathrm{HT}}^{\Gamma_K} = K$ by Theorem 3.1.12 in Chapter II, we need to show $C^{\Gamma_K} = K$.

By Lemma 3.1.11 in Chapter II, we have a Γ_K-equivariant isomorphism

$$B_{\mathrm{HT}} \simeq \mathbb{C}_K[t, t^{-1}] \tag{1.1}$$

where the action of Γ_K on $\mathbb{C}_K[t, t^{-1}]$ is defined by

$$\gamma\left(\sum c_n t^n\right) = \sum \gamma(c_n)\chi(\gamma)^n t^n \qquad \text{for every } \gamma \in \Gamma_K. \tag{1.2}$$

Let us similarly define the action of Γ_K on $\mathbb{C}_K(t)$ and $\mathbb{C}_K((t))$, which respectively denote the field of rational functions and the field of formal Laurent series over $\mathbb{C}_K$. Then the isomorphism (1.1) induces a Γ_K-equivariant injective homomorphism

$$C_{\mathrm{HT}} \simeq \mathbb{C}_K(t) \hookrightarrow \mathbb{C}_K((t)).$$

Hence it suffices to show $\mathbb{C}_K((t))^{\Gamma_K} = K$.

Consider an arbitrary formal Laurent series $p(t) = \sum c_n t^n$ over $\mathbb{C}_K$. Then by (1.2) we have $p(t) \in \mathbb{C}_K((t))^{\Gamma_K}$ if and only if $c_n = \gamma(c_n)\chi(\gamma)^n$ for every $n \in \mathbb{Z}$ and every $\gamma \in \Gamma_K$, or equivalently $c_n \in \mathbb{C}_K(n)^{\Gamma_K}$ for every $n \in \mathbb{Z}$ by Lemma 3.1.11 in Chapter II. We thus obtain the desired assertion by Theorem 3.1.12 in Chapter II.

It remains to check the condition (ii) in Definition 1.1.1. Let $q(t) = \sum d_n t^n$ be an arbitrary nonzero element in $\mathbb{C}_K[t, t^{-1}]$ such that $\mathbb{Q}_p \cdot q(t)$ is stable under the action of Γ_K. We wish to show that $q(t)$ is a unit in $\mathbb{C}_K[t, t^{-1}]$. Since $q(t) \neq 0$, we have $d_m \neq 0$ for some m. It suffices to show that $d_n = 0$ if $n \neq m$.

Let $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ be the character that encodes the action of Γ_K on $\mathbb{Q}_p \cdot q(t)$. Then η is continuous since the action of Γ_K on each $\mathbb{C}_K(n)$ is continuous. In particular, we may consider η as a character with values in $\mathbb{Z}_p^\times$. Now for every $n \in \mathbb{Z}$ and every $\gamma \in \Gamma_K$ we have $\eta(\gamma)\cdot d_n = \gamma(d_n)\chi(\gamma)^n$, or equivalently $d_n = (\eta^{-1}\chi^n)(\gamma)\gamma(d_n)$. This means $d_n \in \mathbb{C}_K(\eta^{-1}\chi^n)^{\Gamma_K}$ for every $n \in \mathbb{Z}$, which implies by Theorem 1.1.8 that $(\eta^{-1}\chi^n)(I_K)$ is finite for any $n \in \mathbb{Z}$ with $d_n \neq 0$.

Suppose for contradiction that we have $d_n \neq 0$ for some $n \neq m$. Our discussion in the preceding paragraph shows that both $\eta^{-1}\chi^n$ and $\eta^{-1}\chi^m$ have finite images on I_K. Hence $\chi^{n-m} = (\eta^{-1}\chi^n) \cdot (\eta^{-1}\chi^m)^{-1}$ also has a finite image, thereby yielding a desired contradiction by Lemma 1.1.7. $\square$

Proposition 1.1.11. *A p-adic representation V of Γ_K is Hodge-Tate if and only if it is B_{HT}-admissible.*

PROOF. By definition we have

$$D_{B_{\mathrm{HT}}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = \bigoplus_{n\in\mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K}. \tag{1.3}$$

Define $\tilde{\alpha}_V$ as in Chapter II, Lemma 3.1.13. Since $\tilde{\alpha}_V$ is injective, it is an isomorphism if and only if the source and the target have the same dimension over $\mathbb{C}_K$, which amounts to the identity $\dim_K D_{B_{\mathrm{HT}}}(V) = \dim_{\mathbb{Q}_p} V$. The desired assertion now follows from definition of Hodge-Tate representations and B_{HT}-admissibility. $\square$

Example 1.1.12. Let V be a p-adic representation of Γ_K which fits into an exact sequence

$$0 \longrightarrow \mathbb{Q}_p(l) \longrightarrow V \longrightarrow \mathbb{Q}_p(m) \longrightarrow 0$$

where l and m are distinct integers. We assert that V is Hodge-Tate. For every $n \in \mathbb{Z}$ we obtain an exact sequence

$$0 \longrightarrow \mathbb{C}_K(l+n) \longrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n) \longrightarrow \mathbb{C}_K(m+n) \longrightarrow 0$$

as $\mathbb{C}_K(n)$ is flat over $\mathbb{Q}_p$, and consequently get a long exact sequence

$$0 \longrightarrow \mathbb{C}_K(l+n)^{\Gamma_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \longrightarrow \mathbb{C}_K(m+n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(l+n)).$$

Then by Theorem 3.1.12 in Chapter II we find

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \cong \begin{cases} K & \text{for } n = -l, -m, \\ 0 & \text{for } n \neq -l, -m. \end{cases}$$

Hence by (1.3) we have

$$\dim_K D_{B_{\mathrm{HT}}}(V) = \sum_{n \in \mathbb{Z}} \dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = 2 = \dim_{\mathbb{Q}_p} V,$$

thereby deducing the desired assertion.

Remark. On the other hand, a self extension of $\mathbb{Q}_p$ may not be Hodge-Tate. For example, the two-dimensional vector space over $\mathbb{Q}_p$ where each $\gamma \in \Gamma_K$ acts as the matrix $\begin{pmatrix} 1 & \log_p(\chi(\gamma)) \\ 0 & 1 \end{pmatrix}$ is not Hodge-Tate. The proof of this statement requires some knowledge about the Sen theory.

Proposition 1.1.13. *Let $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^\times$ be a continuous character. Then $\mathbb{Q}_p(\eta)$ is Hodge-Tate if and only if there exists some $n \in \mathbb{Z}$ such that $(\eta\chi^n)(I_K)$ is finite.*

Proof. Since $\mathbb{Q}_p(\eta)$ is 1-dimensional, Lemma 3.1.13 in Chapter II implies that $\mathbb{Q}_p(\eta)$ is Hodge-Tate if and only if there exists some $n \in \mathbb{Z}$ with $(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \neq 0$, which amounts to the condition $\mathbb{C}_K(\eta\chi^n)^{\Gamma_K} \neq 0$ by Example 1.1.6. We thus obtain the desired assertion by Theorem 1.1.8. ☐

Definition 1.1.14. Let V be a Hodge-Tate representation. We say that an integer $n \in \mathbb{Z}$ is a *Hodge-Tate weight* of V with multiplicity m if we have

$$\dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = m > 0.$$

Example 1.1.15. We record the Hodge-Tate weights for some Hodge-Tate representations.

(1) For every $n \in \mathbb{Z}$ the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is a Hodge-Tate representation with the Hodge-Tate weight $-n$.
(2) For every p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ is a Hodge-Tate representation with the Hodge-Tate weights 0 and -1 by the proof of Corollary 3.4.14 in Chapter II.
(3) For an abelian variety A over K with good reduction, the étale cohomology $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is a Hodge-Tate representation with the Hodge-Tate weights $0, 1, \cdots, n$ by the proof of Corollary 3.4.16 in Chapter II.

Remark. The readers should be aware that many authors use the opposite sign convention for Hodge-Tate weights. We will explain the reason for our choice in §2.4.

1.2. Formal properties of admissible representations

Throughout this subsection, we fix a $(\mathbb{Q}_p, \Gamma_K)$-regular ring B and write $E := B^{\Gamma_K}$.

Theorem 1.2.1. *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ we have the following statements:*

(1) The natural map
$$\alpha_V : D_B(V) \otimes_E B \longrightarrow V \otimes_{\mathbb{Q}_p} B$$
is B-linear, Γ_K-equivariant, and injective.

(2) We have an inequality
$$\dim_E D_B(V) \leq \dim_{\mathbb{Q}_p} V \tag{1.4}$$
with equality if and only if α_V is an isomorphism.

Proof. Let us first consider the statement (1). The natural map α_V is given by
$$\alpha_V : D_B(V) \otimes_E B \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} (B \otimes_E B) \longrightarrow V \otimes_{\mathbb{Q}_p} B,$$
which is B-linear and Γ_K-equivariant by inspection. We need to show that α_V is injective. By Example 1.1.2 the fraction field C of B is $(\mathbb{Q}_p, \Gamma_K)$-regular. We thus have a natural map
$$\beta_V : D_C(V) \otimes_E C \longrightarrow V \otimes_{\mathbb{Q}_p} C$$
which fits into a commutative diagram
$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{\alpha_V} & V \otimes_{\mathbb{Q}_p} B \\ \downarrow & & \downarrow \\ D_C(V) \otimes_E C & \xrightarrow{\beta_V} & V \otimes_{\mathbb{Q}_p} C \end{array}$$
where both vertical maps are injective. Therefore it suffices to prove the injectivity of β_V.

Let (x_i) be a basis of $D_C(V) = (V \otimes_{\mathbb{Q}_p} C)^{\Gamma_K}$ over E. We regard each x_i as an element in $V \otimes_{\mathbb{Q}_p} C$. Note that (x_i) spans $D_C(V) \otimes_E C$ over C.

Assume for contradiction that the kernel of β_V is not trivial. Then we have a nontrivial relation of the form $\sum b_i x_i = 0$ with $b_i \in C$. Let us choose such a relation with minimal length. We may assume $b_r = 1$ for some r. For every $\gamma \in \Gamma_K$ we find
$$0 = \gamma\left(\sum b_i x_i\right) - \sum b_i x_i = \sum (\gamma(b_i) - b_i) x_i.$$
Since the coefficient of x_r vanishes, the minimality of our relation yields $b_i = \gamma(b_i)$ for each b_i, or equivalently $b_i \in C^{\Gamma_K} = E$. Hence our relation gives a nontrivial relation for (x_i) over E, thereby yielding a desired contradiction.

We now proceed to the statement (2). Since the extension of scalars from B to C preserves injectivity, α_V induces an injective map
$$D_B(V) \otimes_E C \hookrightarrow V \otimes_{\mathbb{Q}_p} C. \tag{1.5}$$
The desired inequality (1.4) now follows by observing
$$\dim_C D_B(V) \otimes_E C = \dim_E D_B(V) \qquad \text{and} \qquad \dim_C V \otimes_{\mathbb{Q}_p} C = \dim_{\mathbb{Q}_p} V. \tag{1.6}$$
Hence it remains to consider the equality condition.

If α_V is an isomorphism, the map (1.5) also becomes an isomorphism, thereby yielding equality in (1.4) by (1.6). Let us now assume that equality in (1.4) holds, and write
$$d := \dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

By (1.6) we find that the map (1.5) is an isomorphism for being an injective map between two vector spaces of the same dimension. Let us choose a basis (e_i) of $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ over E and a basis (v_i) of V over $\mathbb{Q}_p$. Then we can represent α_V by a $d \times d$ matrix M_V. We have $\det(M_V) \neq 0$ as α_V induces an isomorphism (1.5). We wish to show $\det(M_V) \in B^\times$. Let us consider the identity

$$\alpha_V(e_1 \wedge \cdots \wedge e_d) = \det(M_V)(v_1 \wedge \cdots \wedge v_d).$$

By construction, Γ_K acts trivially on $e_1 \wedge \cdots \wedge e_d$ and by some $\mathbb{Q}_p$-valued character η on $v_1 \wedge \cdots \wedge v_d$. Since α_V is Γ_K-equivariant, we deduce that Γ_K acts on $\det(M_V)$ by η^{-1}. Hence we obtain $\det(M_V) \in B^\times$ as B is $(\mathbb{Q}_p, \Gamma_K)$-regular, thereby completing the proof. □

Proposition 1.2.2. *The functor D_B is exact and faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$.*

Proof. Let V and W be B-admissible representations. Suppose that $f \in \mathrm{Hom}_{\mathbb{Q}_p[\Gamma_K]}(V, W)$ induces a zero map $D_B(V) \longrightarrow D_B(W)$. Then f induces a zero map $V \otimes_{\mathbb{Q}_p} B \longrightarrow W \otimes_{\mathbb{Q}_p} B$ by Theorem 1.2.1, which means that f must be a zero map. We thus find that the functor D_B is faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$.

It remains to verify that D_B is exact on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$. Let us consider an arbitrary short exact sequence of B-admissible representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

Recall that every algebra over a field is faithfully flat; in particular, B is faithfully flat over both $\mathbb{Q}_p$ and E. Therefore we find that the sequence

$$0 \longrightarrow U \otimes_{\mathbb{Q}_p} B \longrightarrow V \otimes_{\mathbb{Q}_p} B \longrightarrow W \otimes_{\mathbb{Q}_p} B \longrightarrow 0$$

is exact, which implies that the sequence

$$0 \longrightarrow D_B(U) \otimes_E B \longrightarrow D_B(V) \otimes_E B \longrightarrow D_B(W) \otimes_E B \longrightarrow 0$$

is also exact by Theorem 1.2.1. The desired assertion now follows by the fact that B is faithfully flat over E. □

Proposition 1.2.3. *The category $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ is closed under taking subquotients.*

Proof. Consider a short exact sequence of p-adic representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0 \tag{1.7}$$

with $V \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$. We wish to show that both U and W are B-admissible. Since the functor D_B is left exact by construction, we have a left exact sequence

$$0 \longrightarrow D_B(U) \longrightarrow D_B(V) \longrightarrow D_B(W). \tag{1.8}$$

In addition, by Theorem 1.2.1 we have inequalities

$$\dim_E D_B(U) \leq \dim_{\mathbb{Q}_p} U \qquad \text{and} \qquad \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} W. \tag{1.9}$$

Then the exact sequences (1.7) and (1.8) together yield inequalities

$$\dim_E D_B(V) \leq \dim_E D_B(U) + \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} U + \dim_{\mathbb{Q}_p} W = \dim_{\mathbb{Q}_p} V,$$

which are in fact equalities as V is B-admissible. We thus have equalities in (1.9), thereby deducing the desired assertion. □

Remark. However, in general the category $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ is not closed under taking extensions, as noted after Example 1.1.12.

Proposition 1.2.4. *Given* $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$*, we have* $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_{\Gamma_K})$ *with a natural isomorphism*

$$D_B(V) \otimes_E D_B(W) \cong D_B(V \otimes_{\mathbb{Q}_p} W).$$

Proof. By Theorem 1.2.1 we have natural isomorphisms

$$\alpha_V : D_B(V) \otimes_E B \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B \qquad \text{and} \qquad \alpha_W : D_B(W) \otimes_E B \xrightarrow{\sim} W \otimes_{\mathbb{Q}_p} B.$$

Let us consider the natural map

$$D_B(V) \otimes_E D_B(W) \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B) \longrightarrow (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B. \tag{1.10}$$

The image of the first arrow is a Γ_K-invariant space $(V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \otimes (W \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$, while the second arrow is evidently Γ_K-equivariant. Hence we obtain a natural E-linear map

$$D_B(V) \otimes_E D_B(W) \longrightarrow \big((V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B\big)^{\Gamma_K} \cong D_B(V \otimes_{\mathbb{Q}_p} W). \tag{1.11}$$

Moreover, this map is injective since the map (1.10) extends to a B-linear map

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \longrightarrow \big((V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B)\big) \otimes_E B \longrightarrow (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B$$

which coincides with the isomorphism $\alpha_V \otimes \alpha_W$ under the identifications

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \cong (D_B(V)) \otimes_B (D_B(W) \otimes_E B),$$
$$\big((V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B)\big) \otimes_E B \cong (V \otimes_{\mathbb{Q}_p} B \otimes_E B) \otimes_B (W \otimes_{\mathbb{Q}_p} B \otimes_E B),$$
$$(V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B \cong (V \otimes_{\mathbb{Q}_p} B) \otimes_B (W \otimes_{\mathbb{Q}_p} B).$$

Therefore the map (1.11) yields an inequality

$$\dim_E D_B(V \otimes_{\mathbb{Q}_p} W) \geq (\dim_E D(V)) \cdot (\dim_E D_B(W)) = \dim_{\mathbb{Q}_p} V \otimes_{\mathbb{Q}_p} W$$

where the equality follows from the B-admissibility of V and W. We then find that this inequality is indeed an equality by Theorem 1.2.1, thereby deducing that $V \otimes_{\mathbb{Q}_p} W$ is a B-admissible representation with the natural isomorphism (1.11). □

Proposition 1.2.5. *For every* $V \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$*, we have* $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ *and* $\mathrm{Sym}^n V \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ *with natural isomorphisms*

$$\wedge^n(D_B(V)) \cong D_B(\wedge^n(V)) \qquad \textit{and} \qquad \mathrm{Sym}^n(D_B(V)) \cong D_B(\mathrm{Sym}^n(V)).$$

Proof. Let us only consider exterior powers here, as the same argument works with symmetric powers. By Proposition 1.2.4 we have $V^{\otimes n} \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ with a natural isomorphism $D_B(V^{\otimes n}) \cong D_B(V)^{\otimes n}$. Hence by Proposition 1.2.3 we have $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ with a natural E-linear map

$$D_B(V)^{\otimes n} \xrightarrow{\sim} D_B(V^{\otimes n}) \longrightarrow\!\!\!\!\!\rightarrow D_B(\wedge^n(V))$$

where the surjectivity of the second arrow follows from the exactness of D_B as noted in Proposition 1.2.2. It is then straightforward to check that this map factors through the natural surjection $D_B(V)^{\otimes n} \twoheadrightarrow \wedge^n(D_B(V))$. We thus obtain a natural surjective E-linear map

$$\wedge^n(D_B(V)) \longrightarrow\!\!\!\!\!\rightarrow D_B(\wedge^n(V)),$$

which turns out to be an isomorphism since we have

$$\dim_E \wedge^n(D_B(V)) = \dim_E D_B(\wedge^n(V))$$

by the B-admissibility of V and $\wedge^n(V)$. □

Proposition 1.2.6. *For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_K)$, the dual representation $V^\vee$ lies in $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_K)$. Moreover, the natural map*

$$D_B(V) \otimes_E D_B(V^\vee) \cong D_B(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_B(\mathbb{Q}_p) \cong E \tag{1.12}$$

is a perfect pairing.

Proof. Let us first consider the case where $\dim_{\mathbb{Q}_p} V = 1$. We fix a basis vector v for V over $\mathbb{Q}_p$, and denote by $v^\vee$ the corresponding basis vector for $V^\vee$ over $\mathbb{Q}_p$. Then we have a character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ that satisfies

$$\gamma(v) = \eta(\gamma)v \qquad \text{for every } \gamma \in \Gamma_K. \tag{1.13}$$

Since $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ is 1-dimensional over E by the B-admissibility of V, it admits a Γ_K-invariant basis vector $v \otimes b$ for some $b \in B$. Hence by (1.13) we find

$$v \otimes b = \gamma(v \otimes b) = \gamma(v) \otimes \gamma(b) = \eta(\gamma)v \otimes \gamma(b) = v \otimes \eta(\gamma)\gamma(b) \qquad \text{for every } \gamma \in \Gamma_K,$$

or equivalently

$$b = \eta(\gamma)\gamma(b) \qquad \text{for every } \gamma \in \Gamma_K. \tag{1.14}$$

Moreover, we have $b \in B^\times$ as Theorem 1.2.1 yields a natural isomorphism

$$D_B(V) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} B$$

which sends $v \otimes b$ to a basis vector for $V \otimes_{\mathbb{Q}_p} B$ over B. We then find by (1.14) that $D_B(V^\vee) = (V^\vee \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ contains a nonzero vector $v^\vee \otimes b^{-1}$. Hence the inequality

$$\dim_E D_B(V^\vee) \leq \dim_{\mathbb{Q}_p} V^\vee = 1$$

given by Theorem 1.2.1 must be an equality, which implies that $V^\vee$ is B-admissible. We also find that $v^\vee \otimes b^{-1}$ is a basis vector for $D_B(V^\vee)$ over E, and consequently verify that the map (1.12) is a perfect pairing.

We now prove the B-admissibility of $V^\vee$ in the general case. Let us write $d := \dim_{\mathbb{Q}_p} V$. We have a natural Γ_K-equivariant isomorphism

$$\Phi : \det(V^\vee) \otimes_{\mathbb{Q}_p} \wedge^{d-1} V \xrightarrow{\sim} V^\vee$$

such that

$$\Phi : \left((f_1 \wedge \cdots \wedge f_d) \otimes (v_2 \wedge \cdots \wedge v_d)\right)(v_1) = \det(f_i(v_j))$$

for all $f_i \in V^\vee$ and $v_j \in V$. Proposition 1.2.5 implies that both $\det(V) = \wedge^d V$ and $\wedge^{d-1} V$ are B-admissible. Then our discussion in the preceding paragraph shows that $\det(V^\vee) \cong \det(V)^\vee$ is also B-admissible since $\dim_{\mathbb{Q}_p} \det(V) = 1$. Therefore we find that $V^\vee$ is B-admissible by Proposition 1.2.4.

It remains to show that the map (1.12) is a perfect pairing in the general case. Since both V and $V^\vee$ are B-admissible, we have

$$d = \dim_E D_B(V) = \dim_E D_B(V^\vee).$$

Upon choosing bases for $D_B(V)$ and $D_B(V^\vee)$ over E, we can represent the map (1.12) as a $d \times d$ matrix M. Then the map (1.12) is perfect if and only if $\det(M)$ is not zero, or equivalently the induced pairing

$$\det(D_B(V)) \otimes_E \det(D_B(V^\vee)) \longrightarrow E$$

is perfect. We thus deduce the desired assertion from the first paragraph using the identifications

$$\det(D_B(V)) \cong D_B(\det(V)) \qquad \text{and} \qquad \det(D_B(V^\vee)) \cong D_B(\det(V^\vee))$$

given by Proposition 1.2.5. $\square$

2. de Rham representations

The main goal of this section is to define and study the de Rham period ring and de Rham representations. We will use some basic theory of perfectoid fields to provide a modern perspective of Fontaine's original work. Our discussion will introduce many ideas that we will further investigate in Chapter IV. The primary references for this section are Brinon and Conrad's notes [**BC**, §4 and §6] and Scholze's paper [**Sch12**].

2.1. Perfectoid fields and tilting

Definition 2.1.1. Let C be a complete nonarchimedean field of residue characteristic p. We say that C is a *perfectoid field* if it satisfies the following conditions:

(i) The valuation on C is nondiscrete.
(ii) The p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective.

Remark. When we say that a field is nonarchimedean, we always assume that the field is not trivially valued. On the other hand, when we say that a field is valued, we assume that the field may be trivially valued.

Lemma 2.1.2. *Let C be a complete nonarchimedean field of residue characteristic p. Assume that the p-th power map is surjective on C. Then C is a perfectoid field.*

Proof. Let us write ν for the valuation on C. We assert that ν is nondiscrete. Suppose for contradiction that ν is discrete. Take an element $x \in C$ with a minimum positive valuation. Since the p-th power map is surjective on C, we have $x = y^p$ for some $y \in C$. Then we find

$$0 < \nu(y) = \nu(x)/p < \nu(x),$$

thereby obtaining a desired contradiction.

It remains to verify that the p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective. It suffices to show that the p-th power map on $\mathcal{O}_C$ is surjective. Take an arbitrary element $z \in \mathcal{O}_C$. We may write $z = w^p$ for some $w \in C$ as the p-th power map is surjective on C by the assumption. Then we find $w \in \mathcal{O}_C$ by observing

$$\nu(w) = \nu(z)/p > 0.$$

Hence we obtain the desired surjectivity of the p-th power map on $\mathcal{O}_C$. □

Example 2.1.3. Since $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.5, it is a perfectoid field by Lemma 2.1.2.

Proposition 2.1.4. *A nonarchimedean field of characteristic p is perfectoid if and only if it is complete and perfect.*

Proof. By definition, every perfectoid field of characteristic p is complete and perfect. Conversely, every complete nonarchimedean perfect field of characteristic p is perfectoid by Lemma 2.1.2. □

For the rest of this subsection, we let C be a perfectoid field with the valuation ν.

Definition 2.1.5. We define the *tilt* of C by

$$C^\flat := \varprojlim_{x \mapsto x^p} C$$

endowed with the natural multiplication.

A priori, the tilt of C is just a multiplicative monoid. We aim to show that it has a natural structure of a perfectoid field of characteristic p. For every $c = (c_n) \in C^\flat$ we write $c^\sharp := c_0$.

Lemma 2.1.6. *Fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$. Then for arbitrary elements $x, y \in \mathcal{O}_C$ with $x - y \in \varpi\mathcal{O}_C$ we have*

$$x^{p^n} - y^{p^n} \in \varpi^{n+1}\mathcal{O}_C \qquad \text{for each } n = 0, 1, 2, \cdots .$$

Proof. The inequality $\nu(\varpi) \leq \nu(p)$ implies that p is divisible by ϖ in $\mathcal{O}_C$. We also have

$$x^{p^n} - y^{p^n} = \left(y^{p^{n-1}} + (x^{p^{n-1}} - y^{p^{n-1}})\right)^p - y^{p^n} \qquad \text{for each } n = 1, 2, \cdots .$$

Since we have $x - y \in \varpi\mathcal{O}_C$, the desired assertion follows by induction. ☐

Proposition 2.1.7. *For every element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$, the natural projection $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/\varpi\mathcal{O}_C$ induces a multiplicative bijection*

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C.$$

Proof. We wish to construct an inverse

$$\ell : \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C \longrightarrow \varprojlim_{x \mapsto x^p} \mathcal{O}_C.$$

Take an arbitrary element $\overline{c} = (\overline{c}_n) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$. For each n, we choose a lift $c_n \in \mathcal{O}_C$ of $\overline{c}_n$. By construction we have

$$c_{n+m+l}^{p^l} - c_{n+m} \in \varpi\mathcal{O}_C \qquad \text{for all } l, m, n \geq 0,$$

and consequently find

$$c_{n+m+l}^{p^{m+l}} - c_{n+m}^{p^m} \in \varpi^{m+1}\mathcal{O}_C \qquad \text{for all } n, m \geq 0$$

by Lemma 2.1.6. Hence for each $n \geq 0$ the sequence $(c_{n+m}^{p^m})_{m \geq 0}$ converges in $\mathcal{O}_C$ for being Cauchy. In addition, the limit does not depend on the choice of the c_n's by Lemma 2.1.6. Let us now write

$$\ell_n(\overline{c}) := \lim_{m \to \infty} c_{n+m}^{p^m} \qquad \text{for each } n \geq 0.$$

We then obtain the desired inverse by setting

$$\ell(\overline{c}) := (\ell_n(\overline{c})) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C,$$

thereby completing the proof. ☐

Proposition 2.1.8. *The tilt $C^\flat$ of C is naturally a complete valued field of characteristic p with the valuation $\nu^\flat$ given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$. Moreover, the valuation ring of $C^\flat$ is given by*

$$\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C.$$

Proof. Fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$. The ring $\mathcal{O}_C/\varpi\mathcal{O}_C$ is of characteristic p since ϖ divides p in $\mathcal{O}_C$ by construction. Hence the ring structure on $\mathcal{O}_C/\varpi\mathcal{O}_C$ induces a natural ring structure on $\varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$, which in turn yields a ring structure on

$$\mathcal{O} := \varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C \tag{2.1}$$

where the isomorphism is given by Proposition 2.1.7. Moreover, this ring structure on $\mathcal{O}$ does not depend on the choice of ϖ; indeed, by the proof of Proposition 2.1.7 we find that the sum of two arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $\mathcal{O}$ is given by

$$(a+b)_n = \lim_{m\to\infty} (a_{m+n} + b_{m+n})^{p^m}.$$

We then identify $C^\flat$ as the fraction field of $\mathcal{O}$. It is clear by construction that $C^\flat$ is perfect of characteristic p.

We assert that $C^\flat$ admits a valuation $\nu^\flat$ given by $\nu^\flat(c) := \nu(c^\sharp)$ for every $c \in C^\flat$. It is evident by construction that $\nu^\flat$ is a multiplicative homomorphism. Let us now consider arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $C^\flat$. We wish to establish an inequality

$$\nu^\flat(a + b) \geq \min(\nu^\flat(a), \nu^\flat(b)).$$

We may assume $\nu^\flat(a) \geq \nu^\flat(b)$, or equivalently $\nu(a_0) \geq \nu(b_0)$. Then for each $n \geq 0$ we have

$$\nu(a_n) = \frac{1}{p^n}\nu(a_0) \geq \frac{1}{p^n}\nu(b_0) = \nu(b_n),$$

which means $a_n/b_n \in \mathcal{O}_C$. Therefore we may write $a = br$ for some $r \in \mathcal{O}$ and find

$$\nu^\flat(a+b) = \nu^\flat((r+1)b) = \nu^\flat(r+1) + \nu^\flat(b) \geq \nu^\flat(b) = \min(\nu^\flat(a), \nu^\flat(b))$$

where the inequality follows by observing $r + 1 \in \mathcal{O}$.

Let us now take an arbitrary element $c = (c_n) \in C^\flat$. We have an inequality

$$\nu(c_n) = \frac{1}{p^n}\nu(c_0) = \frac{1}{p^n}\nu^\flat(c) \qquad \text{for each } n \geq 0. \tag{2.2}$$

Hence we deduce that $\mathcal{O}$ is indeed the valuation ring of $C^\flat$. Moreover, given any $N > 0$ the inequality (2.2) implies that we have $\nu(c_n) \geq \nu(\varpi)$ for all $n \leq N$ if and only if $\nu^\flat(c) \geq p^N\nu(\varpi)$. Therefore the bijection (2.1) becomes a homeomorphism if we endow $\mathcal{O}_{C^\flat} = \mathcal{O}$ and $\varprojlim\limits_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ respectively with the $\nu^\flat$-adic topology and the inverse limit topology. As the latter topology is complete, it follows that $C^\flat$ is complete. □

Remark. Our proof of Proposition 2.1.8 remains valid if C is replaced by an arbitrary complete nonarchimedean field L (with its "tilt" $L^\flat$ defined as in Definition 2.1.5). However, if L is not perfectoid the valuation on the tilt $L^\flat$ becomes trivial. For example, the "tilt" of $\mathbb{Q}_p$ is easily seen to be isomorphic to $\mathbb{F}_p$ (with the trivial valuation).

Proposition 2.1.9. *The map $\mathcal{O}_{C^\flat} \longrightarrow \mathcal{O}_C/p\mathcal{O}_C$ which sends each $c \in \mathcal{O}_{C^\flat}$ to the image of $c^\sharp$ in $\mathcal{O}_C/p\mathcal{O}_C$ is a ring homomorphism.*

Proof. This is evident by the definition of the natural ring structure on $\mathcal{O}_{C^\flat}$ given in the proof of Proposition 2.1.8. □

Lemma 2.1.10. *For every $y \in \mathcal{O}_C$ there exists an element $z \in \mathcal{O}_{C^\flat}$ with $y - z^\sharp \in p\mathcal{O}_C$.*

Proof. Let $\overline{y}$ denote the image of y in $\mathcal{O}_C/p\mathcal{O}_C$. Since the p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective, there exists an element $z' = (z'_n) \in \varprojlim\limits_{x\mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C$ with $z'_0 = \overline{x}$. The assertion now follows by taking $z \in \mathcal{O}_{C^\flat}$ to be the image of z' under the bijection

$$\mathcal{O}_{C^\flat} \simeq \varprojlim_{x\mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C$$

as given by Proposition 2.1.7 and Proposition 2.1.8. □

Proposition 2.1.11. *The valued fields C and $C^\flat$ have the same value groups.*

PROOF. Let $\nu^\flat$ be the valuation on $C^\flat$ given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$. Since we have $\nu^\flat\left((C^\flat)^\times\right) \subseteq \nu(C^\times)$ by construction, we only need to show $\nu(C^\times) \subseteq \nu^\flat\left((C^\flat)^\times\right)$. Let us consider an arbitrary element $y \in C^\times$. We wish to find an element $z \in (C^\flat)^\times$ with $\nu^\flat(z) = \nu(y)$. Since ν is nondiscrete, we can choose an element $\varpi \in \mathcal{O}_C$ with $0 < \nu(\varpi) < \nu(p)$. Let us write $y = \varpi^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_C$ with $\nu(u) < \nu(\varpi)$. By Lemma 2.1.10 there exist elements $\varpi^\flat$ and $u^\flat$ in $\mathcal{O}_{C^\flat}$ with $\varpi - (\varpi^\flat)^\sharp \in p\mathcal{O}_C$ and $u - (u^\flat)^\sharp \in p\mathcal{O}_C$. Then we find

$$\nu^\flat(\varpi^\flat) = \nu((\varpi^\flat)^\sharp) = \nu\left((\varpi) - (\varpi - (\varpi^\flat)^\sharp)\right) = \nu(\varpi),$$
$$\nu^\flat(u^\flat) = \nu((u^\flat)^\sharp) = \nu\left((u) - (u - (u^\flat)^\sharp)\right) = \nu(u).$$

Hence we obtain the desired assertion by taking $z = (\varpi^\flat)^n u^\flat$. □

Corollary 2.1.12. *The field $C^\flat$ is a perfectoid field of characteristic p.*

PROOF. Proposition 2.1.11 implies that the value group of $C^\flat$ is not trivial. Since $C^\flat$ is perfect by construction, the assertion follows by Proposition 2.1.4 and Proposition 2.1.8. □

Corollary 2.1.13. *If C is of characteristic p, there exists a natural identification $C^\flat \cong C$.*

PROOF. As C is perfect by Proposition 2.1.4, the assertion is evident by construction. □

Example 2.1.14. Let $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ denote the p-adic completions of $\bigcup_{n\geq 1} \mathbb{Q}_p(p^{1/p^n})$. The p-adic valuation on $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is clearly not discrete. In addition, the valuation ring of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is easily seen to be $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$, the p-adic completion of the $\mathbb{Z}_p$-algebra obtained by adjoining all p-power roots of p. We also have an isomorphism

$$\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}/p \simeq \mathbb{Z}_p[p^{1/p^\infty}]/p \simeq \mathbb{F}_p[u^{1/p^\infty}]/u$$

where $\mathbb{F}_p[u^{1/p^\infty}]$ denotes the perfection of the polynomial ring $\mathbb{F}_p[u]$. Since the p-th power map on $\mathbb{F}_p[u^{1/p^\infty}]/u$ is evidently surjective, we deduce that $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is a perfectoid field. Moreover, we obtain an identification

$$\varprojlim_{x\mapsto x^p} \widehat{\mathbb{Z}_p[p^{1/p^\infty}]}/p \simeq \varprojlim_{x\mapsto x^p} \mathbb{F}_p[u^{1/p^\infty}]/u \simeq \widehat{\mathbb{F}_p[u^{1/p^\infty}]}$$

where $\widehat{\mathbb{F}_p[u^{1/p^\infty}]}$ denotes the u-adic completion of $\mathbb{F}_p[u^{1/p^\infty}]$, and consequently find that the tilt of $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ is isomorphic to $\widehat{\mathbb{F}_p((u^{1/p^\infty}))}$, the u-adic completion of the perfection of the Laurent series ring $\mathbb{F}_p((u))$.

Remark. A similar argument shows that the p-adic completion of $\mathbb{Q}_p(\mu_{p^\infty})$ is also a perfectoid field whose tilt is isomorphic to $\mathbb{F}_p((u^{1/p^\infty}))$. Therefore the tilting functor is not fully faithful on the category of perfectoid fields over $\mathbb{Q}_p$. This fact is a foundation for Scholze's theory of *diamonds* as developed in [**Sch18**] and [**SW20**].

On the other hand, for every perfectoid field C the tilting functor induces an equivalence between the category of perfectoid fields over C and the category of perfectoid fields over $C^\flat$. This is a special case of the *tilting equivalence*, which is the main result of Scholze's paper [**Sch12**].

2.2. The de Rham period ring B_{dR}

For the rest of this chapter, we write $F := \mathbb{C}_K^\flat$ for the tilt of $\mathbb{C}_K$. In addition, for every element $c = (c_n)_{n\geq 0}$ in F we write $c^\sharp := c_0$. We also fix a valuation ν on $\mathbb{C}_K$ with $\nu(p) = 1$, and let $\nu^\flat$ denote the valuation on F given by $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in F$.

Definition 2.2.1. We define the *infinitesimal period ring*, denoted by A_{inf}, to be the ring of Witt vectors over $\mathcal{O}_F$. For every $c \in \mathcal{O}_F$, we write $[c]$ for its Teichmüller lift in A_{inf}.

Remark. The ring A_{inf} is not $(\mathbb{Q}_p, \Gamma_K)$-regular in any meaningful way.

Proposition 2.2.2. *There exists a surjective ring homomorphism $\theta : A_{\mathrm{inf}} \longrightarrow \mathcal{O}_{\mathbb{C}_K}$ with*

$$\theta\left(\sum_{n=0}^{\infty}[c_n]p^n\right) = \sum_{n=0}^{\infty} c_n^\sharp p^n \qquad \text{for all } c_n \in \mathcal{O}_F. \tag{2.3}$$

Proof. Let us define a map $\overline{\theta} : \mathcal{O}_F \longrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ by

$$\overline{\theta}(c) = \overline{c^\sharp} \qquad \text{for every } c \in \mathcal{O}_F$$

where $\overline{c^\sharp}$ denotes the image of $c^\sharp$ in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Then $\overline{\theta}$ is a ring homomorphism as noted in Proposition 2.1.9. Moreover, by construction $\overline{\theta}$ lifts to a map $\hat{\theta} : \mathcal{O}_F \longrightarrow \mathcal{O}_{\mathbb{C}_K}$ defined by

$$\hat{\theta}(c) = c^\sharp \qquad \text{for every } c \in \mathcal{O}_F.$$

Since $\hat{\theta}$ is clearly multiplicative, Lemma 2.3.1 from Chapter II yields a ring homomorphism $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ satisfying (2.3).

It remains to establish the surjectivity of θ. Let x be an arbitrary element in $\mathcal{O}_{\mathbb{C}_K}$. Since $\mathcal{O}_{\mathbb{C}_K}$ is p-adically complete, it is enough to find elements $c_0, c_1, \cdots \in \mathcal{O}_F$ with

$$x - \sum_{n=0}^{m} c_n^\sharp p^n \in p^{m+1}\mathcal{O}_{\mathbb{C}_K} \qquad \text{for each } m = 0, 1, \cdots.$$

In fact, by Lemma 2.1.10 we can inductively define each c_m to be any element in $\mathcal{O}_F$ with

$$\frac{1}{p^m}\left(x - \sum_{n=0}^{m-1} c_n^\sharp p^n\right) - c_m^\sharp \in p\mathcal{O}_{\mathbb{C}_K},$$

thereby completing the proof. □

Remark. As explained in [**BC**, Lemma 4.4.1], it is possible to construct the homomorphism θ in Proposition 2.2.2 without using Lemma 2.3.1 from Chapter II. In this approach, we first define θ as a set theoretic map given by (2.3), then show that θ is indeed a ring homomorphism using the explicit addition and multiplication rules for A_{inf}.

For the rest of this chapter, we let $\theta : A_{\mathrm{inf}} \longrightarrow \mathcal{O}_{\mathbb{C}_K}$ be the ring homomorphism constructed in Proposition 2.2.2, and let $\theta[1/p] : A_{\mathrm{inf}}[1/p] \longrightarrow \mathbb{C}_K$ be the induced map on $A_{\mathrm{inf}}[1/p]$. We also choose an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$, and set $\xi := [p^\flat] - p \in A_{\mathrm{inf}}$.

Definition 2.2.3. We define the *de Rham local ring* by

$$B_{\mathrm{dR}}^+ := \varprojlim_j A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^j.$$

We denote by θ_{dR}^+ the natural projection $B_{\mathrm{dR}}^+ \twoheadrightarrow A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])$.

Remark. We will soon define the de Rham period ring B_{dR} as the fraction field of B_{dR}^+ after verifying that B_{dR}^+ is a discrete valuation ring. At this point, it is instructive to explain Fontaine's insight behind the construction of B_{dR}. As briefly discussed in Chapter I, the main motivation for constructing the de Rham period ring B_{dR} is to obtain the de Rham comparison isomorphism as stated in Chapter I, Theorem 1.2.2. Recall that the de Rham cohomology admits a canonical filtration, called the *Hodge filtration*, whose associated graded vector space recovers the Hodge cohomology. Since the Hodge-Tate decomposition can be stated in terms of the Hodge-Tate period ring B_{HT} as noted after Theorem 1.2.1 in Chapter I, Fontaine sought to construct B_{dR} as a ring with a canonical filtration which recovers B_{HT} as the associated graded algebra. His idea was to construct the subring B_{dR}^+ as a complete discrete valuation ring with an action of Γ_K such that there exist Γ_K-equivariant isomorphisms

$$B_{\text{dR}}^+/\mathfrak{m}_{\text{dR}} \simeq \mathbb{C}_K \qquad \text{and} \qquad \mathfrak{m}_{\text{dR}}/\mathfrak{m}_{\text{dR}}^2 \simeq \mathbb{C}_K(1)$$

where $\mathfrak{m}_{\text{dR}}$ denotes the maximal ideal of B_{dR}^+. In characteristic p, the theory of Witt vectors provides a natural way to construct a complete discrete valuation ring with a specified perfect residue field. Fontaine judiciously applied the Witt vector construction to the field $\mathbb{C}_K$ of characteristic 0 by passing to characteristic p. More precisely, he first defined the ring A_{inf} as the ring of Witt vectors over the perfect ring

$$R_K := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K},$$

which he called the *perfection* of $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$, then constructed the homomorphism $\theta[1/p]$ as above to realize $\mathbb{C}_K$ as a quotient of $A_{\text{inf}}[1/p]$; indeed, since R_K is naturally isomorphic to $\mathcal{O}_F$ by Proposition 2.1.7, our construction provides a modern interpretation for the construction of R_K and A_{inf}. Fontaine then define B_{dR}^+ as the completion of $A_{\text{inf}}[1/p]$ with respect to $\ker(\theta[1/p])$ as in Definition 2.2.3, and showed that B_{dR}^+ satisfies all the desired properties.

We now aim to show that B_{dR}^+ is a complete discrete valuation ring with $\mathbb{C}_K$ as the residue field. To this end we study several properties of $\ker(\theta)$.

Lemma 2.2.4. *For each $n \geq 0$ we have* $\ker(\theta) \cap p^n A_{\text{inf}} = p^n \ker(\theta)$.

Proof. We only need to show $\ker(\theta) \cap p^n A_{\text{inf}} \subseteq p^n \ker(\theta)$ since the reverse containment is obvious. Let a be an arbitrary element in $\ker(\theta) \cap p^n A_{\text{inf}}$. We may write $a = p^n b$ for some $b \in A_{\text{inf}}$. Then we have

$$0 = \theta(a) = \theta(p^n b) = p^n \theta(b),$$

and consequently find $\theta(b) = 0$ since $\mathcal{O}_{\mathbb{C}_K}$ has no nonzero p-torsion. We thus deduce that $a = p^n b$ lies in $p^n \ker(\theta)$. □

Lemma 2.2.5. *Every element $a \in \ker(\theta)$ is of the form $a = c\xi + dp$ for some $c, d \in A_{\text{inf}}$.*

Proof. We wish to show that a lies in the ideal generated by ξ and p, or equivalently by $[p^\flat]$ and p. Let us write

$$a = \sum_{n \geq 0} [c_n] p^n = [c_0] + \sum_{n \geq 1} [c_n] p^n \qquad \text{for some } c_n \in \mathcal{O}_F.$$

It suffices to show that $[c_0]$ is divisible by $[p^\flat]$. Since we have $0 = \theta(a) = \sum_{n \geq 0} c_n^\sharp p^n$, we deduce that $c_0^\sharp$ is divisible by p, and consequently find

$$\nu^\flat(c_0) = \nu(c_0^\sharp) \geq \nu(p) = \nu((p^\flat)^\sharp) = \nu^\flat(p^\flat).$$

Hence there exists some $r \in \mathcal{O}_F$ with $c_0 = p^\flat r$, which yields $[c_0] = [p^\flat][r]$ as desired. □

Proposition 2.2.6. *The ideal* $\ker(\theta)$ *in* A_{inf} *is generated by* ξ.

PROOF. By definition we have

$$\theta(\xi) = \theta([p^\flat] - p) = (p^\flat)^\sharp - p = p - p = 0.$$

Hence we only need to show that $\ker(\theta)$ lies in the ideal ξA_{inf}. Let a be an arbitrary element in $\ker(\theta)$. Since A_{inf} is p-adically separated and complete by construction, it suffices to show that there exist elements $c_0, c_1, \cdots, \in A_{\text{inf}}$ with

$$a - \sum_{n=0}^{m} c_n \xi p^n \in p^{m+1} A_{\text{inf}} \qquad \text{for each } m \geq 0.$$

We proceed by induction on m to find such $c_0, c_1, \cdots \in A_{\text{inf}}$. As both ξ and a lie in $\ker(\theta)$, we have

$$a - \sum_{n=0}^{m-1} c_n \xi p^n \in \ker(\theta) \cap p^m A_{\text{inf}} = p^m \ker(\theta)$$

by the induction hypothesis and Lemma 2.2.4. Then by Lemma 2.2.5 we find some $c_m, d_m \in A_{\text{inf}}$ with

$$a - \sum_{n=0}^{m-1} c_n \xi p^n = p^m (c_m \xi + p d_m),$$

or equivalently

$$a - \sum_{n=0}^{m} c_n \xi p^n = p^{m+1} d_m$$

as desired. □

Remark. Proposition 2.2.6 yields an isomorphism of valuation rings $A_{\text{inf}}/\xi A_{\text{inf}} \simeq \mathcal{O}_{\mathbb{C}_K}$. Since the construction of A_{inf} depends only on the field F, we find that the ideal ξA_{inf} contains all necessary information for recovering the perfectoid field $\mathbb{C}_K$ from its tilt F. In fact, as we will see in Chapter IV, every "untilt" of F can be realized as the fraction field of A_{inf}/I for a unique principal ideal I in A_{inf}.

Corollary 2.2.7. *The ideal* $\ker(\theta[1/p])$ *in* $A_{\text{inf}}[1/p]$ *is generated by* ξ.

PROOF. For every $a \in \ker(\theta[1/p])$, we have $p^n a \in \ker(\theta)$ for some $n > 0$. Hence the assertion follows from Proposition 2.2.6. □

Remark. In fact, our proof shows that every generator of $\ker(\theta)$ generates $\ker(\theta[1/p])$.

Lemma 2.2.8. *Every* $a \in A_{\text{inf}}[1/p]$ *with* $\xi a \in A_{\text{inf}}$ *is an element in* A_{inf}.

PROOF. Since we have $\theta(\xi a) = \theta[1/p](\xi a) = 0$, there exists an element $b \in A_{\text{inf}}$ with $\xi a = \xi b$ by Proposition 2.2.6. We then find $a = b$ as A_{inf} is an integral domain, thereby deducing the desired assertion. □

Lemma 2.2.9. *For all* $j \geq 1$ *we have* $A_{\text{inf}} \cap \ker(\theta[1/p])^j = \ker(\theta)^j$.

PROOF. We only need to show $A_{\text{inf}} \cap \ker(\theta[1/p])^j \subseteq \ker(\theta)^j$ since the reverse containment is obvious. Let a be an arbitrary element in $A_{\text{inf}} \cap \ker(\theta[1/p])^j$. Corollary 2.2.7 implies that there exists some $r \in A_{\text{inf}}[1/p]$ with $a = \xi^j b$. Then we find $b \in A_{\text{inf}}$ by Lemma 2.2.8, and consequently obtain $a \in \ker(\theta)^j$ by Proposition 2.2.6. □

Proposition 2.2.10. *We have* $\bigcap_{j=1}^{\infty} \ker(\theta)^j = \bigcap_{j=1}^{\infty} \ker(\theta[1/p])^j = 0.$

Proof. By Lemma 2.2.9 we find

$$\bigcap_{j=1}^{\infty} \ker(\theta[1/p])^j = \left(\bigcap_{j=1}^{\infty} \ker(\theta)^j \right) [1/p]. \tag{2.4}$$

Hence it suffices to prove $\bigcap_{j=1}^{\infty} \ker(\theta)^j = 0$. Take an arbitrary element $c \in \bigcap_{j=1}^{\infty} \ker(\theta)^j$. As usual, let us write $c = \sum [c_n] p^n$ for some $c_n \in \mathcal{O}_F$. By Proposition 2.2.6 we find that c is divisible by arbitrarily high powers of $\xi = [p^\flat] - p$. This implies that c_0 is divisible by arbitrarily high powers of $p^\flat$, which in turn means $c_0 = 0$ as we have

$$\nu^\flat(p^\flat) = \nu((p^\flat)^\sharp) = \nu(p) = 1 > 0.$$

Hence we find some $c' \in A_{\inf}$ with $c = pc'$. Moreover, Lemma 2.2.9 and (2.4) together yield

$$c' \in A_{\inf} \cap \left(\bigcap_{j=1}^{\infty} \ker(\theta)^j \right) [1/p] = A_{\inf} \cap \left(\bigcap_{j=1}^{\infty} \ker(\theta[1/p])^j \right) = \bigcap_{j=1}^{\infty} \ker(\theta)^j.$$

Then an easy induction shows that c is infinitely divisible by p, which in turn implies $c = 0$ as $A_{\inf}$ is p-adically complete. □

Corollary 2.2.11. *The natural map*

$$A_{\inf}[1/p] \longrightarrow \varprojlim_j A_{\inf}[1/p]/\ker(\theta[1/p])^j = B_{\mathrm{dR}}^+$$

is injective. In particular, we may canonically identify $A_{\inf}[1/p]$ *as a subring of* B_{dR}^+.

Proposition 2.2.12. *The ring* B_{dR}^+ *is a complete discrete valuation ring with* $\ker(\theta_{\mathrm{dR}}^+)$ *as the maximal ideal and* $\mathbb{C}_K$ *as the residue field. Moreover, the element* ξ *is a uniformizer of* B_{dR}^+.

Proof. Since both θ_{dR}^+ and $\theta[1/p]$ are surjective by construction, we have an isomorphism

$$B_{\mathrm{dR}}^+/\ker(\theta_{\mathrm{dR}}^+) \simeq A_{\inf}[1/p]/\ker(\theta[1/p]) \simeq \mathbb{C}_K.$$

In addition, a general fact as stated in [**Sta**, Tag 05GI] implies that every element $b \in B_{\mathrm{dR}}^+$ is a unit if and only if $\theta_{\mathrm{dR}}^+(b)$ is a unit in $B_{\mathrm{dR}}^+/\ker(\theta_{\mathrm{dR}}^+) \simeq \mathbb{C}_K$, or equivalently $b \notin \ker(\theta_{\mathrm{dR}}^+)$. Therefore B_{dR}^+ is a local ring with $\ker(\theta_{\mathrm{dR}}^+)$ as the maximal ideal and $\mathbb{C}_K$ as the residue field.

Consider an arbitrary nonzero element $b \in B_{\mathrm{dR}}^+$. For each $j \geq 0$, let b_j and ξ_j respectively denote the image of b and ξ under the projection $B_{\mathrm{dR}}^+ \twoheadrightarrow A_{\inf}[1/p]/\ker(\theta[1/p])^j$. Take the maximum $i \geq 0$ with $b_i = 0$. Then for each $j > i$ we have

$$b_j \in \ker(\theta[1/p])^i/\ker(\theta)^j \qquad \text{and} \qquad b_j \notin \ker(\theta[1/p])^{i+1}/\ker(\theta)^j.$$

Hence by Proposition 2.2.6 we may write $b_j = \xi_j^i u_j$ for some $u_j \in B_{\mathrm{dR}}^+/\ker(\theta[1/p])^j$ with $u_j \notin \ker(\theta[1/p])/\ker(\theta[1/p])^j$. For each $j > i$ we let u_j' denote the image of u_j in $B_{\mathrm{dR}}^+/\ker(\theta[1/p])^{j-i}$. By construction the sequence $(u_j')_{j>i}$ gives rise to a unit $u \in B_{\mathrm{dR}}^+$ such that $b = \xi^i u$. Moreover, it is not hard to see that u is uniquely determined by b even though the u_j's are not uniquely determined. We thus deduce that B_{dR}^+ is a discrete valuation ring with ξ as a uniformizer. The completeness of B_{dR}^+ then follows by Proposition 2.2.6 and Proposition 2.2.10. □

Definition 2.2.13. We define the *de Rham period ring* B_{dR} as the fraction field of B_{dR}^+.

Remark. Our argument so far in this subsection remains valid if $\mathbb{C}_K$ is replaced by any algebraically closed perfectoid field of characteristic 0. Hence we may regard B_{dR} as a functor from the category of algebraically closed perfectoid fields over $\mathbb{Q}_p$ to the category of complete valued fields. In Chapter IV we will provide a geometric interpretation of this statement.

Proposition 2.2.14. *For every uniformizer π of B_{dR}^+, the filtration $\{\pi^n B_{\mathrm{dR}}^+\}_{n\in\mathbb{Z}}$ of B_{dR} satisfies the following properties:*

(i) $\pi^{n+1} B_{\mathrm{dR}}^+ \subseteq \pi^n B_{\mathrm{dR}}^+$ *for all* $n \in \mathbb{Z}$.

(ii) $\bigcap_{n\in\mathbb{Z}} \pi^n B_{\mathrm{dR}}^+ = 0$ *and* $\bigcup_{n\in\mathbb{Z}} \pi^n B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}$.

(iii) $(\pi^m B_{\mathrm{dR}}^+)\cdot(\pi^n B_{\mathrm{dR}}^+) \subseteq \pi^{m+n} B_{\mathrm{dR}}^+$ *for all* $m, n \in \mathbb{Z}$.

PROOF. This is immediate by Proposition 2.2.12. □

Remark. The filtration $\{\pi^i B_{\mathrm{dR}}^+\}_{n\in\mathbb{Z}}$ does not depend on the choice of π; indeed, we have an identification $\pi^i B_{\mathrm{dR}}^+ = \ker(\theta_{\mathrm{dR}}^+)^n$ for each $n \in \mathbb{Z}$.

Proposition 2.2.15. *Let $W(k)$ denote the ring of Witt vectors over k, and let K_0 denote the fraction field of $W(k)$.*

(1) *The field K is a finite totally ramified extension of K_0.*

(2) *There exists a natural commutative diagram*

$$\begin{array}{ccc} K_0 & \longrightarrow & A_{\mathrm{inf}}[1/p] \\ \big\downarrow & & \big\downarrow \\ \overline{K} & \hookrightarrow & B_{\mathrm{dR}}^+ \\ & \searrow & \big\downarrow \theta_{\mathrm{dR}}^+ \\ & & \mathbb{C}_K \end{array} \tag{2.5}$$

where the diagonal map is the natural inclusion.

PROOF. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}_K$. The natural projection $\mathcal{O}_K/p\mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/\mathfrak{m} = k$ admits a canonical section $s : k \longrightarrow \mathcal{O}_K/p\mathcal{O}_K$; indeed, the ring $\mathcal{O}_K/p\mathcal{O}_K$ is a vector space over k with basis given by $1, \pi, \cdots, \pi^{e-1}$, where π is a uniformizer in $\mathcal{O}_K$ with $\nu(\pi) = 1/e$. In addition, the map s induces a homomorphism of discretely valued fields $K_0 \longrightarrow K$ by Lemma 2.3.1 from Chapter II. We thus obtain the statement (1) by observing that both K_0 and K are complete with the residue field k.

Let us now prove the statement (2). Since k is perfect, the section $s : k \longrightarrow \mathcal{O}_K/p\mathcal{O}_K$ induces a natural map

$$k \longrightarrow \varprojlim_{x\to x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \cong \mathcal{O}_F$$

where the isomorphism is given by Proposition 2.1.7. We then obtain the top horizontal arrow in (2.5) by Lemma 2.3.1 from Chapter II, and the upper right vertical arrow in (2.5) by Corollary 2.2.11. Hence B_{dR}^+ is a complete discrete valuation ring over K_0. Moreover, the statement (1) implies that $\overline{K}$ is a separable algebraic extension of K_0, thereby yielding the left vertical map in (2.5). Now we deduce by Hensel's lemma that the subfield $\overline{K}$ of the residue field $\mathbb{C}_K$ uniquely lifts to a subfield of B_{dR}^+ over K_0, thereby obtaining the middle horizontal arrow in (2.5). □

Our final goal of this subsection is to describe and study the natural action of Γ_K on B_{dR}, especially in relation to the natural filtration on B_{dR} as described in Proposition 2.2.14. We invoke the following technical result without proof.

Proposition 2.2.16. *There exists a refinement of the discrete valuation topology on B_{dR}^+ that satisfies the following properties:*

(i) The natural map $A_{\mathrm{inf}} \longrightarrow B_{\mathrm{dR}}^+$ identifies A_{inf} as a closed subring of B_{dR}^+.
(ii) The map $\theta[1/p]$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.
(iii) There exists a continuous map $\log : \mathbb{Z}_p(1) \longrightarrow B_{\mathrm{dR}}^+$ with

$$\log(\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \qquad \text{for every } \varepsilon \in \mathbb{Z}_p(1)$$

under the natural identification $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \left\{ \varepsilon \in \mathcal{O}_F : \varepsilon^\sharp = 1 \right\}$.
(iv) The multiplication by any uniformizer yields a closed embedding on B_{dR}^+.
(v) The ring B_{dR}^+ is complete.

Remark. We will eventually prove Proposition 2.2.16 in Chapter IV after constructing the Fargues-Fontaine curve. There will be no circular reasoning; the construction of the Fargues-Fontaine curve relies only on results that we have discussed prior to Proposition 2.2.16. The readers can also find a sketch of the proof in [**BC**, Exercise 4.5.3] and the discussion after Definition 4.4.7 in loc. cit.

Here we provide an indication on why Proposition 2.2.16 is necessary for our discussion. As we will soon describe, the natural Γ_K-action on B_{dR} is induced by the action of Γ_K on $\mathbb{C}_K$ such that the map θ_{dR}^+ is Γ_K-equivariant. Proposition 2.2.16 ensures that the map θ_{dR}^+ is furthermore continuous with respect to the p-adic topology on $\mathbb{C}_K$, thereby allowing us to exploit the topological properties of the Γ_K-action on $\mathbb{C}_K$.

For the rest of this chapter, we consider the map $\log : \mathbb{Z}_p(1) \longrightarrow B_{\mathrm{dR}}^+$ as given by Proposition 2.2.16. In addition, we fix a $\mathbb{Z}_p$-basis element $\varepsilon \in \mathbb{Z}_p(1)$ and write $t := \log(\varepsilon)$. We often regard ε as an element in $\mathcal{O}_F$ via the identification $\mathbb{Z}_p(1) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$ as noted in Proposition 2.2.16. We also regard $A_{\mathrm{inf}}[1/p]$ as a subring of B_{dR}^+ in light of Corollary 2.2.11.

Lemma 2.2.17. *We have $\nu^\flat(\varepsilon - 1) = \dfrac{p}{p-1}$.*

Proof. By construction we may write $\varepsilon = (\zeta_{p^n})$ where each ζ_{p^n} is a primitive p^n-th root of unity in $\overline{K}$. Then we find

$$\begin{aligned} \nu^\flat(\varepsilon - 1) = \nu\left((\varepsilon-1)^\sharp\right) &= \nu\left(\lim_{n\to\infty} (\zeta_{p^n} - 1)^{p^n}\right) \\ &= \lim_{n\to\infty} p^n \nu(\zeta_{p^n} - 1) = \lim_{n\to\infty} \frac{p^n}{p^{n-1}(p-1)} \\ &= \frac{p}{p-1} \end{aligned}$$

by the proof of Proposition 2.1.8 and the continuity of the valuation ν. □

Lemma 2.2.18. *The element ξ divides $[\varepsilon] - 1$ in A_{inf}.*

Proof. By construction we have

$$\theta([\varepsilon] - 1) = \varepsilon^\sharp - 1 = 1 - 1 = 0.$$

Hence the assertion immediately follows from Proposition 2.2.6. □

Proposition 2.2.19. *The element $t \in B_{\mathrm{dR}}^+$ is a uniformizer.*

PROOF. By Lemma 2.2.18 we have

$$[\varepsilon] - 1 \in \xi A_{\mathrm{inf}} \qquad \text{and} \qquad t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in \xi B_{\mathrm{dR}}^+.$$

We also find $\dfrac{([\varepsilon]-1)^n}{n} \in \xi^2 B_{\mathrm{dR}}^+$ for each $n \geq 2$. Since ξ is a uniformizer of B_{dR}^+ as noted in Proposition 2.2.12, it suffices to prove $[\varepsilon] - 1 \notin \xi^2 B_{\mathrm{dR}}^+$.

Suppose for contradiction that $[\varepsilon] - 1$ lies in $\xi^2 B_{\mathrm{dR}}^+$. Then the proof of Proposition 2.2.12 shows that the image of $[\varepsilon] - 1$ under the projection $B_{\mathrm{dR}}^+ \twoheadrightarrow A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^2$ is zero. Since $[\varepsilon] - 1$ is an element of A_{inf}, we find $[\varepsilon] - 1 \in \ker(\theta[1/p])^2 \cap A_{\mathrm{inf}}$. Hence Proposition 2.2.6 and Lemma 2.2.9 together imply that $[\varepsilon] - 1$ is divisible by ξ^2 in A_{inf}.

Since the first coefficients in the Teichmüller expansions for $[\varepsilon] - 1$ and ξ^2 are respectively equal to $[\varepsilon - 1]$ and $[(p^\flat)^2]$, we obtain

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^2) = 2\nu^\flat(p^\flat) = 2\nu((p^\flat)^\sharp) = 2\nu(p) = 2.$$

On the other hand, if p is odd we have $\nu^\flat(\varepsilon - 1) < 2$ by Lemma 2.2.17. Therefore we find $p = 2$. Let us now take an element $c \in A_{\mathrm{inf}}$ with $[\varepsilon] - 1 = \xi^2 c$. We then compare the coefficients of p in the Teichmüller expansions of both sides and find $\varepsilon - 1 = c_1 (p^\flat)^4$ where c_1 denote the coefficient of p in the Teichmüller expansion of c. Hence we have

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^4) = 4\nu^\flat(p^\flat) = 4\nu((p^\flat)^\sharp) = 4\nu(p) = 4,$$

thereby obtaining a desired contradiction since Lemma 2.2.17 yields $\nu^\flat(\varepsilon - 1) = 2$. □

Remark. Our proof shows that the power series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n}$ converges with respect to the discrete valuation topology on B_{dR}^+. Hence we can define the uniformizer $t \in B_{\mathrm{dR}}^+$ without using the topology given in Proposition 2.2.16.

Lemma 2.2.20. *For every $m \in \mathbb{Z}_p$ we have $\log(\varepsilon^m) = m \log(\varepsilon)$.*

PROOF. Let us first consider the case where m is an integer. We know that the identity

$$\log((1+x)^m) = m \log(1+x)$$

holds as formal power series. Since $t = \log(\varepsilon)$ converges in B_{dR}^+ as noted Proposition 2.2.19, we set $x = \varepsilon - 1$ to obtain the desired assertion.

We now consider the general case. Let us choose a sequence (m_i) of integers such that $m_i - m$ is divisible by p^i. As $\log(\varepsilon) = t$ is a uniformizer of B_{dR}^+ by Proposition 2.2.19, we find

$$\lim_{i \to \infty} m_i \log(\varepsilon) = m \log(\varepsilon)$$

by Proposition 2.2.16. In addition, it is straightforward to verify

$$\lim_{i \to \infty} \varepsilon^{m_i} = \varepsilon^m$$

with respect to the valuation topology on F. We thus find

$$\log(\varepsilon^m) = \log\left(\lim_{i \to \infty} \varepsilon^{m_i}\right) = \lim_{i \to \infty} \log(\varepsilon^{m_i}) = \lim_{i \to \infty} m_i \log(\varepsilon) = m \log(\varepsilon)$$

where the second identity follows from the continuity of the logarithm map as noted in Proposition 2.2.16. □

Theorem 2.2.21 (Fontaine [**Fon82**]). *The natural action of Γ_K on B_{dR} satisfies the following properties:*

(i) The logarithm map and θ_{dR}^+ are Γ_K-equivariant.
(ii) For every $\gamma \in \Gamma_K$ we have $\gamma(t) = \chi(\gamma)t$.
(iii) Each $t^n B_{\mathrm{dR}}^+$ is stable under the action of Γ_K.
(iv) There exists a canonical Γ_K-equivariant isomorphism

$$\bigoplus_{n\in\mathbb{Z}} t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_K(n) = B_{\mathrm{HT}}.$$

(v) B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular with a natural identification $B_{\mathrm{dR}}^{\Gamma_K} \cong K$.

Proof. Let us first describe the natural action of Γ_K on B_{dR}. The action of Γ_K on $\mathbb{C}_K$ naturally induces an action on $F = \varprojlim_{x\mapsto x^p} \mathbb{C}_K$ as the p-th power map on $\mathbb{C}_K$ is Γ_K-equivariant. More precisely, given an arbitrary element $x = (x_n) \in F$ we have $\gamma(x) = (\gamma(x_n))$ for every $\gamma \in \Gamma_K$. It is then evident that $\mathcal{O}_F$ is stable under the action of Γ_K. Hence by functoriality of Witt vectors we obtain a natural action of Γ_K on $A_{\mathrm{inf}}[1/p]$ with

$$\gamma\left(\sum [c_n]p^n\right) = \sum [\gamma(c_n)]p^n \qquad \text{for all } \gamma \in \Gamma_K, c_n \in \mathcal{O}_F.$$

We then find that θ and $\theta[1/p]$ are both Γ_K-equivariant by construction, and consequently deduce that both $\ker(\theta)$ and $\ker(\theta[1/p])$ are stable under the action of Γ_K. Therefore Γ_K naturally acts on $B_{\mathrm{dR}}^+ = \varprojlim_j A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^j$ and its fraction field B_{dR}.

With our discussion in the preceding paragraph, it is straightforward to verify the property (i). Moreover, for every $\gamma \in \Gamma_K$ we use Lemma 2.2.20 to find

$$\gamma(t) = \gamma(\log(\varepsilon)) = \log(\gamma(\varepsilon)) = \log(\varepsilon^{\chi(\gamma)}) = \chi(\gamma)\log(\varepsilon) = \chi(\gamma)t,$$

thereby deducing the property (ii). The property (iii) then immediately follows as B_{dR}^+ is stable under the action of Γ_K.

Let us now prove the property (iv). We note that the natural isomorphism

$$B_{\mathrm{dR}}^+/\ker(\theta_{\mathrm{dR}}^+) = B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ \simeq A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p]) \simeq \mathbb{C}_K.$$

is Γ_K-equivariant, and consequently obtain Γ_K-equivariant isomorphisms

$$\ker(\theta_{\mathrm{dR}}^+)^n/\ker(\theta_{\mathrm{dR}}^+)^{n+1} = t^nB_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \simeq \mathbb{C}_K(n) \qquad \text{for all } n \in \mathbb{Z}$$

by the property (ii) and Lemma 3.1.11 in Chapter II. These isomorphisms are canonical since t is uniquely determined up to $\mathbb{Z}_p^\times$-multiple by Lemma 2.2.20. We thus obtain the desired Γ_K-equivariant isomorphism by taking the direct sum of the above isomorphisms.

It remains to verify the property (v). The field B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular as noted in Example 1.1.2. In addition, since the map θ_{dR}^+ is Γ_K-equivariant by construction, the natural injective homomorphism $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ given by Proposition 2.2.15 is also Γ_K-equivariant, thereby inducing an injective homomorphism

$$K = \overline{K}^{\Gamma_K} \hookrightarrow (B_{\mathrm{dR}}^+)^{\Gamma_K} \hookrightarrow B_{\mathrm{dR}}^{\Gamma_K}. \tag{2.6}$$

Then by the properties (iii) and (iv) we get an injective K-algebra homomorphism

$$\bigoplus_{n\in\mathbb{Z}} (B_{\mathrm{dR}}^{\Gamma_K} \cap t^nB_{\mathrm{dR}}^+)/(B_{\mathrm{dR}}^{\Gamma_K} \cap t^{n+1}B_{\mathrm{dR}}^+) \hookrightarrow B_{\mathrm{HT}}^{\Gamma_K}.$$

Since we have $B_{\mathrm{HT}}^{\Gamma_K} \cong K$ by Theorem 3.1.12 in Chapter II, the K-algebra on the source has dimension at most 1. Hence we find $\dim_K B_{\mathrm{dR}}^{\Gamma_K} \leq 1$, thereby completing the proof by (2.6) □

2.3. Filtered vector spaces

In this subsection we set up a categorical framework for our discussion of B_{dR}-admissible representations in the next subsection.

Definition 2.3.1. Let L be an arbitrary field.

(1) A *filtered vector space* over L is a vector space V over L along with a collection of subspaces $\{ \mathrm{Fil}^n(V) \}_{n\in\mathbb{Z}}$ that satisfies the following properties:
  (i) $\mathrm{Fil}^n(V) \supseteq \mathrm{Fil}^{n+1}(V)$ for every $n \in \mathbb{Z}$.
  (ii) $\displaystyle\bigcap_{n\in\mathbb{Z}} \mathrm{Fil}^n(V) = 0$ and $\displaystyle\bigcup_{n\in\mathbb{Z}} \mathrm{Fil}^n(V) = V$.

(2) A *graded vector space* over L is a vector space V over L along with a direct sum decomposition $V = \displaystyle\bigoplus_{n\in\mathbb{Z}} V_n$.

(3) A L-linear map between two filtered vector spaces V and W over L is called a *morphism of filtered vector spaces* if it maps each $\mathrm{Fil}^n(V)$ into $\mathrm{Fil}^n(W)$.

(4) A L-linear map between two graded vector spaces $V = \displaystyle\bigoplus_{n\in\mathbb{Z}} V_n$ and $W = \displaystyle\bigoplus_{n\in Z} W_n$ over L is called a *morphism of graded vector spaces* if it maps each V_n into W_n.

(5) For a filtered vector space V over L, we define its *associated graded vector space* by
$$\mathrm{gr}(V) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Fil}^n(V)/\,\mathrm{Fil}^{n+1}(V)$$
and write $\mathrm{gr}^n(V) := \mathrm{Fil}^n(V)/\,\mathrm{Fil}^{n+1}(V)$ for every $n \in \mathbb{Z}$.

(6) We denote by Fil_L the category of finite dimensional filtered vector spaces over L.

Example 2.3.2. We present some motivating examples for our discussion.

(1) The ring B_{dR} is a filtered K-algebra with $\mathrm{Fil}^n(B_{\mathrm{dR}}) := t^n B_{\mathrm{dR}}^+$ and $\mathrm{gr}(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$ by Proposition 2.2.14 and Theorem 2.2.21.

(2) For a proper smooth variety X over K, the de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$ with the Hodge filtration is a filtered vector space over K whose associated graded vector space recovers the Hodge cohomology.

(3) For every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, we may regard $D_{B_{\mathrm{dR}}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ as a filtered vector space over K with
$$\mathrm{Fil}^n(D_{B_{\mathrm{dR}}}(V)) := (V \otimes_{\mathbb{Q}_p} t^n B_{\mathrm{dR}}^+)^{\Gamma_K}.$$

Remark. For an arbitrary proper smooth variety X over K, we have a canonical Γ_K-equivariant isomorphism of filtered vector spaces
$$D_{B_{\mathrm{dR}}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K)$$
by Theorem 1.2.2 in Chapter I. In particular, we can recover the Hodge filtration on $H^n_{\mathrm{dR}}(X/K)$ from the Γ_K-action on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$.

Lemma 2.3.3. *Let V be a finite dimensional filtered vector space over a field L. There exists a basis $(v_{i,j})$ for V such that for every $n \in \mathbb{Z}$ the vectors $v_{i,j}$ with $i \geq n$ form a basis for $\mathrm{Fil}^n(V)$.*

PROOF. Since V is finite dimensional, we have $\mathrm{Fil}^n(V) = 0$ for all sufficiently large n and $\mathrm{Fil}^n(V) = 0$ for all sufficiently small n. Hence we can construct such a basis by inductively extending a basis for $\mathrm{Fil}^n(V)$ to a basis for $\mathrm{Fil}^{n-1}(V)$. □

Definition 2.3.4. Let L be an arbitrary field.

(1) Given two filtered vector spaces V and W over L, we define the *convolution filtration* on $V \otimes_L W$ by

$$\mathrm{Fil}^n(V \otimes_L W) := \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes_L \mathrm{Fil}^j(W).$$

(2) For every filtered vector space V over L, we define the *dual filtration* on the dual space $V^\vee = \mathrm{Hom}_L(V, L)$ by

$$\mathrm{Fil}^n(V^\vee) := \left\{ f \in V^\vee : \mathrm{Fil}^{1-n}(V) \subseteq \ker(f) \right\}.$$

(3) We define the *unit object* $L[0]$ in Fil_L to be the vector space L with the filtration

$$\mathrm{Fil}^n(L[0]) := \begin{cases} L & \text{if } n \leq 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Remark. The use of $\mathrm{Fil}^{1-n}(V)$ rather than $\mathrm{Fil}^{-n}(V)$ in (2) is to ensure that $L[0]$ is self-dual.

Proposition 2.3.5. *Let V be a filtered vector space over a field L. Then we have canonical isomorphisms of filtered vector spaces*

$$V \otimes_L L[0] \cong L[0] \otimes_L V \cong V \qquad \textit{and} \qquad (V^\vee)^\vee \cong V.$$

Proof. For every $n \in \mathbb{Z}$ we find

$$\mathrm{Fil}^n(V \otimes_L L[0]) = \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes_L \mathrm{Fil}^j(L[0]) \cong \sum_{i \geq n} \mathrm{Fil}^i(V) = \mathrm{Fil}^n(V),$$

and consequently obtain an identification of filtered vector spaces

$$V \otimes_L L[0] \cong L[0] \otimes_L V \cong V.$$

Moreover, the natural evaluation isomorphism $\epsilon : V \cong (V^\vee)^\vee$ yields an isomorphism of filtered vector spaces since for every $n \in \mathbb{Z}$ we have

$$\begin{aligned} \mathrm{Fil}^n\left((V^\vee)^\vee\right) &\cong \left\{ v \in V : \mathrm{Fil}^{1-n}(V^\vee) \subseteq \ker(\epsilon(v)) \right\} \\ &= \left\{ v \in V : f(v) = 0 \text{ for all } f \in \mathrm{Fil}^{1-n}(V^\vee) \right\} \\ &= \left\{ v \in V : f(v) = 0 \text{ for all } f \in V^\vee \text{ with } \mathrm{Fil}^n(V) \subseteq \ker(f) \right\} \\ &= \mathrm{Fil}^n(V). \end{aligned}$$

Therefore we complete the proof. □

Proposition 2.3.6. *Let V and W be finite dimensional filtered vector spaces over a field L. Then we have a natural identification of filtered vector spaces*

$$(V \otimes_L W)^\vee \cong V^\vee \otimes_L W^\vee.$$

Proof. By Lemma 2.3.3 we can choose bases $(v_{i,k})$ and $(w_{j,l})$ for V and W such that for every $n \in \mathbb{Z}$ the vectors $(v_{i,k})_{i \geq n}$ and $(w_{j,l})_{j \geq n}$ respectively form bases for $\mathrm{Fil}^n(V)$ and $\mathrm{Fil}^n(W)$. Let $(f_{i,k})$ and $(g_{j,l})$ be the dual bases for $V^\vee$ and $W^\vee$. Then the vectors $(f_{i,k} \otimes g_{j,l})$ form a basis for the vector space $(V \otimes_L W)^\vee \cong V^\vee \otimes_L W^\vee$. Moreover, for every $n \in \mathbb{Z}$ the vectors $(f_{i,k})_{i \leq -n}$ and $(g_{j,l})_{j \leq -n}$ respectively form bases for $\mathrm{Fil}^n(V^\vee)$ and $\mathrm{Fil}^n(W^\vee)$. Hence we find that for every $n \in \mathbb{Z}$ both $\mathrm{Fil}^n\left((V \otimes_L W)^\vee\right)$ and $\mathrm{Fil}^n(V^\vee \otimes_L W^\vee)$ are spanned by the vectors $(f_{i,k} \otimes g_{j,l})_{i+j \leq -n}$, thereby deducing the desired assertion. □

Lemma 2.3.7. *Let* $V = \bigoplus_{n\in\mathbb{Z}} V_n$ *and* $W = \bigoplus_{n\in\mathbb{Z}} W_n$ *be graded vector spaces over a field* L. *A morphism* $f : V \longrightarrow W$ *of graded vector spaces is an isomorphism if and only if it is bijective.*

Proof. The assertion immediately follows by observing that f is the direct sum of the induced morphisms $f_n : V_n \longrightarrow W_n$. □

Proposition 2.3.8. *Let* L *be an arbitrary field. A bijective morphism* $f : V \longrightarrow W$ *in* Fil_L *is an isomorphism in* Fil_L *if and only if the induced map* $\mathrm{gr}(f) : \mathrm{gr}(V) \longrightarrow \mathrm{gr}(W)$ *is bijective.*

Proof. If f is an isomorphism of filtered vector spaces, then $\mathrm{gr}(f)$ is clearly an isomorphism. Let us now assume that $\mathrm{gr}(f)$ is an isomorphism. We wish to show that for every $n \in \mathbb{Z}$ the induced map $\mathrm{Fil}^n(f) : \mathrm{Fil}^n(V) \longrightarrow \mathrm{Fil}^n(W)$ is an isomorphism. Since each $\mathrm{Fil}^n(f)$ is injective by the bijectivity of f, it suffices to show

$$\dim_L \mathrm{Fil}^n(V) = \dim_L \mathrm{Fil}^n(W) \qquad \text{for every } n \in \mathbb{Z}.$$

The map $\mathrm{gr}(f)$ is an isomorphism of graded vector spaces by Lemma 2.3.7, and consequently induces an isomorphism

$$\mathrm{gr}^n(V) \simeq \mathrm{gr}^n(W) \qquad \text{for every } n \in \mathbb{Z}.$$

Hence for every $n \in \mathbb{Z}$ we find

$$\dim_L \mathrm{Fil}^n(V) = \sum_{i\geq n} \dim_L \mathrm{gr}^i(V) = \sum_{i\geq n} \dim_L \mathrm{gr}^i(W) = \dim_L \mathrm{Fil}^n(W)$$

as desired. □

Example 2.3.9. Let us define $L[1]$ to be the vector space L with the filtration

$$\mathrm{Fil}^n(L[1]) := \begin{cases} L & \text{if } n \leq 1, \\ 0 & \text{if } n > 1. \end{cases}$$

The bijective morphism $L[0] \longrightarrow L[1]$ given by the identity map on L is not an isomorphism in Fil_L since $\mathrm{Fil}^1(L[0]) = 0$ and $\mathrm{Fil}^1(L[1]) = L$ are not isomorphic. Moreover, the induced map $\mathrm{gr}(L[0]) \longrightarrow \mathrm{gr}(L[1])$ is a zero map.

Proposition 2.3.10. *Let* L *be an arbitrary field. For any* $V, W \in \mathrm{Fil}_L$ *there exists a natural isomorphism of graded vector spaces*

$$\mathrm{gr}(V \otimes_L W) \cong \mathrm{gr}(V) \otimes_L \mathrm{gr}(W).$$

Proof. Since we have a direct sum decomposition

$$\mathrm{gr}(V) \otimes_L \mathrm{gr}(W) = \bigoplus_{n\in\mathbb{Z}} \left(\bigoplus_{i+j=n} \mathrm{gr}^i(V) \otimes_L \mathrm{gr}^j(W) \right),$$

it suffices to find a natural isomorphism

$$\mathrm{gr}^n(V \otimes_L W) \cong \bigoplus_{i+j=n} \mathrm{gr}^i(V) \otimes_L \mathrm{gr}^j(W) \qquad \text{for every } n \in \mathbb{Z}. \tag{2.7}$$

By Lemma 2.3.3 we can choose bases $(v_{i,k})$ and $(w_{j,l})$ for V and W such that for every $n \in \mathbb{Z}$ the vectors $(v_{i,k})_{i\geq n}$ and $(w_{j,l})_{j\geq n}$ respectively span $\mathrm{Fil}^n(V)$ and $\mathrm{Fil}^n(W)$. Let $\overline{v}_{i,k}$ denote the image of $v_{i,k}$ under the map $\mathrm{Fil}^i(V) \twoheadrightarrow \mathrm{gr}^i(V)$, and let $\overline{w}_{j,l}$ denote the image of $w_{j,l}$ under the map $\mathrm{Fil}^j(W) \twoheadrightarrow \mathrm{gr}^j(W)$. Since each $\mathrm{Fil}^n(V \otimes_L W)$ is spanned by the vectors $(v_{i,k} \otimes w_{j,l})_{i+j\geq n}$, we obtain the identification (2.7) by observing that both sides are spanned by the vectors $(\overline{v}_{i,k} \otimes \overline{w}_{j,l})_{i+j=n}$. □

2.4. Properties of de Rham representations

Definition 2.4.1. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *de Rham* if it is B_{dR}-admissible. We write $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K) := \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{dR}}}(\Gamma_K)$ for the category of de Rham p-adic Γ_K-representations. In addition, we write D_{HT} and D_{dR} respectively for the functors $D_{B_{\mathrm{HT}}}$ and $D_{B_{\mathrm{dR}}}$.

Example 2.4.2. Below are some important examples of de Rham representations.

(1) For every $n \in \mathbb{Z}$ the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is de Rham; indeed, the inequality
$$\dim_K D_{\mathrm{dR}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$
given by Theorem 1.2.1 is an equality, as $D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$ by Theorem 2.2.21.

(2) Every $\mathbb{C}_K$-admissible representation is de Rham by a result of Sen.

(3) For every proper smooth variety X over K, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham by a theorem of Faltings as briefly discussed in Chapter I, Theorem 1.2.2.

The general formalism discussed in §1 readily yields a number of nice properties for de Rham representations and the functor D_{dR}. Our main goal in this subsection is to extend these properties in order to incorporate the additional structures induced by the filtration $\{\, t^n B_{\mathrm{dR}}^+ \,\}_{n \in \mathbb{Z}}$ on B_{dR}.

Lemma 2.4.3. *Given any $n \in \mathbb{Z}$, every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is de Rham if and only if $V(n)$ is de Rham.*

Proof. Since we have identifications
$$V(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n) \qquad \text{and} \qquad V \cong V(n) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-n),$$
the assertion follows from Proposition 1.2.4 and the fact that every Tate twist of $\mathbb{Q}_p$ is de Rham as noted in Example 2.4.2. □

Proposition 2.4.4. *Let V be a de Rham representation of Γ_K. Then V is Hodge-Tate with a natural K-linear isomorphism of graded vector spaces*
$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V).$$

Proof. For every $n \in \mathbb{Z}$ we have a short exact sequence
$$0 \longrightarrow t^{n+1}B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \longrightarrow 0,$$
which induces an exact sequence
$$0 \longrightarrow \left(V \otimes_{\mathbb{Q}_p} t^{n+1}B_{\mathrm{dR}}^+\right)^{\Gamma_K} \longrightarrow \left(V \otimes_{\mathbb{Q}_p} t^n B_{\mathrm{dR}}^+\right)^{\Gamma_K} \longrightarrow \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+)\right)^{\Gamma_K}$$
and consequently yields an injective K-linear map
$$\mathrm{gr}^n(D_{\mathrm{dR}}(V)) = \mathrm{Fil}^n(D_{\mathrm{dR}}(V))/\,\mathrm{Fil}^{n+1}(D_{\mathrm{dR}}(V)) \hookrightarrow \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+)\right)^{\Gamma_K}.$$
Therefore we obtain an injective K-linear map of graded vector spaces
$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow \bigoplus_{n \in \mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+)\right)^{\Gamma_K} \cong (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = D_{\mathrm{HT}}(V)$$
where the middle isomorphism follows from Theorem 2.2.21. We then find
$$\dim_K D_{\mathrm{dR}}(V) = \dim_K \mathrm{gr}(D_{\mathrm{dR}}(V)) \leq \dim_K D_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$$
where the last inequality follows from Theorem 1.2.1. Since V is de Rham, both inequalities should be in fact equalities, thereby yielding the desired assertion. □

Example 2.4.5. Let V be an extension of $\mathbb{Q}_p(m)$ by $\mathbb{Q}_p(n)$ with $m < n$. We assert that V is de Rham. By Lemma 2.4.3 we may assume $m = 0$. Then we have a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \longrightarrow V \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{2.8}$$

Since the functor D_{dR} is left exact by construction, we obtain a left exact sequence

$$0 \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p(n)) \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p).$$

We wish to show $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V = 2$. Since we have

$$\dim_K D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = \dim_K D_{\mathrm{dR}}(\mathbb{Q}_p) = 1$$

by Example 2.4.2, it suffices to show the surjectivity of the map $D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K$.

As B_{dR}^+ is faithfully flat over $\mathbb{Q}_p$, the sequence (2.8) yields a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow 0.$$

In addition, by Theorem 2.2.21 and Proposition 2.2.15 we have identifications

$$(\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (t^n B_{\mathrm{dR}}^+)^{\Gamma_K} = 0,$$
$$(\mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (B_{\mathrm{dR}}^+)^{\Gamma_K} \cong K.$$

We thus obtain a long exact sequence

$$0 \longrightarrow 0 \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow K \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+).$$

Since we have $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \subseteq (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K} = D_{\mathrm{dR}}(V)$, it suffices to prove

$$H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+) = 0. \tag{2.9}$$

By Theorem 2.2.21 we have a short exact sequence

$$0 \longrightarrow t^{n+1} B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+ \longrightarrow \mathbb{C}_K(n) \longrightarrow 0,$$

which in turn yields a long exact sequence

$$\mathbb{C}_K(n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(n)).$$

Then by Theorem 3.1.12 in Chapter II we obtain an identification

$$H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \cong H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+). \tag{2.10}$$

Hence by induction we only need to prove (2.9) for $n = 1$.

Take an arbitrary element $\alpha_1 \in H^1(\Gamma_K, t B_{\mathrm{dR}}^+)$. We wish to show $\alpha_1 = 0$. Regarding α_1 as a cocycle, we use (2.10) to inductively construct sequences (α_m) and (y_m) with the following properties:

(i) $\alpha_m \in H^1(\Gamma_K, t^m B_{\mathrm{dR}}^+)$ and $y_m \in t^m B_{\mathrm{dR}}^+$ for all $m \geq 1$,
(ii) $\alpha_{m+1}(\gamma) = \alpha_m(\gamma) + \gamma(y_m) - y_m$ for all $\gamma \in \Gamma_K$ and $m \geq 1$.

Now, since t is a uniformizer in B_{dR}^+ as noted in Proposition 2.2.19, we may take an element $y = \sum y_m \in B_{\mathrm{dR}}^+$. Then we have

$$\alpha_1(\gamma) + \gamma(y) - y \in H^1(\Gamma_K, t^m B_{\mathrm{dR}}^+) \qquad \text{for all } \gamma \in \Gamma_K \text{ and } m \geq 0,$$

and consequently find $\alpha_1(\gamma) + \gamma(y) - y = 0$ for all $\gamma \in \Gamma_K$. We thus deduce $\alpha_1 = 0$ as desired.

Remark. It is a highly nontrivial fact that every non-splitting extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is Hodge-Tate but not de Rham. The existence of such an extension follows from the identification

$$\mathrm{Ext}^1_{\mathbb{Q}_p[\Gamma_K]}(\mathbb{Q}_p(1), \mathbb{Q}_p) \cong H^1(\Gamma_K, \mathbb{Q}_p(-1)) \cong K$$

where the second isomorphism is a consequence of the Tate local duality for p-adic representations. Moreover, such an extension is Hodge-Tate as noted in Example 1.1.12. The difficult part is to prove that such an extension is not de Rham. For this part we need a very deep result that every de Rham representation is *potentially semistable*.

Proposition 2.4.6. *Let V be a de Rham representation of Γ_K. For every $n \in \mathbb{Z}$ we have $\mathrm{gr}^n(D_{\mathrm{dR}}(V)) \neq 0$ if and only if n is a Hodge-Tate weight of V.*

PROOF. This is an immediate consequence of Proposition 2.4.4 and Definition 1.1.14. $\square$

Remark. Proposition 2.4.6 provides the main reason for our choice of the sign convention in the definition of Hodge-Tate weights. In fact, under our convention the Hodge-Tate weights of a de Rham representation V indicate where the filtration of $D_{\mathrm{dR}}(V)$ has a jump. In particular, for a proper smooth variety X over K, the Hodge-Tate weights of the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ give the positions of "jumps" for the Hodge filtration on the de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$ by the isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K).$$

Example 2.4.7. The Tate twist $\mathbb{Q}_p(m)$ of $\mathbb{Q}_p$ is a 1-dimensional de Rham representation with the Hodge-Tate weight $-m$ as noted in Example 1.1.15 and Example 2.4.2. Hence by Proposition 2.4.6 we find

$$\mathrm{Fil}^n(D_{\mathrm{dR}}(\mathbb{Q}_p(m))) \cong \begin{cases} K & \text{for } n \leq -m, \\ 0 & \text{for } n > -m. \end{cases}$$

In particular, for $m = 0$ we obtain an identification $D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K[0]$.

Proposition 2.4.8. *For every $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(\Gamma_K)$, we have a natural Γ_K-equivariant isomorphism of filtered vector spaces*

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

PROOF. Since V is de Rham, Theorem 1.2.1 implies that the natural map

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{dR}} \otimes_K B_{\mathrm{dR}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

is a Γ_K-equivariant isomorphism of vector spaces over B_{dR}. Moreover, this map is a morphism of filtered vector spaces as each arrow above is easily seen to be a morphism of filtered vector spaces. Hence by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \longrightarrow \mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \tag{2.11}$$

is an isomorphism. By Proposition 2.3.10, Proposition 2.4.4 and Theorem 2.2.21 we obtain identifications

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \otimes_K \mathrm{gr}(B_{\mathrm{dR}}) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}},$$
$$\mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{dR}}) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}.$$

We thus identify the map (2.11) with the natural map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1. The desired assertion now follows by Proposition 2.4.4. $\square$

Proposition 2.4.9. *The functor D_{dR} with values in Fil_K is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$.*

Proof. Let Vec_K denote the category of finite dimensional vector spaces over K. The faithfulness of D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ is an immediate consequence of Proposition 1.2.2 since the forgetful functor $\mathrm{Fil}_K \longrightarrow \mathrm{Vec}_K$ is faithful. Hence it remains to verify the exactness of D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$. Consider an exact sequence of de Rham representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0. \tag{2.12}$$

The functor D_{dR} with values in Fil_K is left exact by construction. In other words, for every $n \in \mathbb{Z}$ we have a left exact sequence

$$0 \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(W)). \tag{2.13}$$

We wish to show that this sequence extends to a short exact sequence. By Proposition 1.2.2 the sequence (2.12) induces a short exact sequence of vector spaces

$$0 \longrightarrow D_{\mathrm{HT}}(U) \longrightarrow D_{\mathrm{HT}}(V) \longrightarrow D_{\mathrm{HT}}(W) \longrightarrow 0.$$

Moreover, by the definition of D_{HT} we find that this sequence is indeed a short exact sequence of graded vector spaces. Then by Proposition 2.4.4 we may rewrite this sequence as

$$0 \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(U)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(W)) \longrightarrow 0.$$

by Proposition 2.4.4. Hence for every $n \in \mathbb{Z}$ we have

$$\begin{aligned}\dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) &= \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V)) \\ &= \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(U)) + \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(W)) \\ &= \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) + \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(W)),\end{aligned}$$

thereby deducing that the sequence (2.13) extends to a short exact sequence as desired. □

Corollary 2.4.10. *Let V be a de Rham representation. Every subquotient W of V is a de Rham representation with $D_{\mathrm{dR}}(W)$ naturally identified as a subquotient of $D_{\mathrm{dR}}(V)$ in Fil_K.*

Proof. This is an immediate consequence of Proposition 1.2.3 and Proposition 2.4.9. □

Proposition 2.4.11. *Given any $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with a natural isomorphism of filtered vector spaces*

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W) \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W). \tag{2.14}$$

Proof. By Proposition 1.2.4 we find $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ and obtain the desired isomorphism (2.14) as a map of vector spaces. Moreover, since the construction of the map (2.14) rests on the multiplicative structure of B_{dR} as shown in the proof of Proposition 1.2.4, it is straightforward to verify that the map (2.14) is a morphism in Fil_K. Hence by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)) \tag{2.15}$$

is an isomorphism. Since both V and W are Hodge-Tate by Proposition 2.4.4, we have a natural isomorphism

$$D_{\mathrm{HT}}(V) \otimes_K D_{\mathrm{HT}}(W) \cong D_{\mathrm{HT}}(V \otimes_{\mathbb{Q}_p} W) \tag{2.16}$$

by Proposition 1.2.4. Therefore we complete the proof by identifying the maps (2.15) and (2.16) using Proposition 2.3.10 and Proposition 2.4.4. □

Proposition 2.4.12. *For every de Rham representation V, we have $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ and $\mathrm{Sym}^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with natural isomorphisms of filtered vector spaces*

$$\wedge^n(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{dR}}(\wedge^n(V)) \qquad \text{and} \qquad \mathrm{Sym}^n(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{dR}}(\mathrm{Sym}^n(V)).$$

PROOF. Proposition 1.2.5 implies that both $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are de Rham for every $n \geq 1$. In addition, Proposition 1.2.5 yields the desired isomorphisms as maps of vector spaces. Then Corollary 2.4.10 and Proposition 2.4.11 together imply that these maps are isomorphisms in Fil_K. □

Proposition 2.4.13. *For every de Rham representation V, the dual representation $V^\vee$ is de Rham with a natural perfect paring of filtered vector spaces*

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(V^\vee) \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K[0]. \qquad (2.17)$$

PROOF. By Proposition 1.2.6 we find $V^\vee \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ and obtain the desired perfect pairing as a map of vector spaces. Moreover, Proposition 2.4.11 implies that this pairing is a morphism in Fil_K. We thus obtain a bijective morphism of filtered vector spaces

$$D_{\mathrm{dR}}(V)^\vee \longrightarrow D_{\mathrm{dR}}(V^\vee).$$

Therefore by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V^\vee)) \qquad (2.18)$$

is an isomorphism. Since V is Hodge-Tate by Proposition 2.4.4, we have a natural isomorphism

$$D_{\mathrm{HT}}(V)^\vee \cong D_{\mathrm{HT}}(V^\vee) \qquad (2.19)$$

by Proposition 1.2.6. We thus deduce the desired assertion by identifying the maps (2.18) and (2.19) using Proposition 2.4.4. □

Let us now discuss some additional facts about de Rham representations and the functor D_{dR}.

Proposition 2.4.14. *Let V be a p-adic representation of Γ_K. Let L be a finite extension of K with absolute Galois group Γ_L.*

(1) There exists a natural isomorphism of filtered vector spaces

$$D_{\mathrm{dR},K}(V) \otimes_K L \cong D_{\mathrm{dR},L}(V)$$

where we set $D_{\mathrm{dR},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ and $D_{\mathrm{dR},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_L}$.

(2) V is de Rham if and only if it is de Rham as a representation of Γ_L.

PROOF. We only need to prove the first statement, as the second statement immediately follows from the first statement. Let L' be the Galois closure of L over K with the absolute Galois group $\Gamma_{L'}$ and set $D_{\mathrm{dR},L'}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_{L'}}$. Then we have identifications

$$D_{\mathrm{dR},K}(V) \otimes_K L = (D_{\mathrm{dR},K}(V) \otimes_K L')^{\mathrm{Gal}(L'/L)} \qquad \text{and} \qquad D_{\mathrm{dR},L}(V) = D_{\mathrm{dR},L'}(V)^{\mathrm{Gal}(L'/L)}.$$

Hence we may replace L by L' to assume that L is Galois over K. Moreover, since the construction of B_{dR} depends only on $\mathbb{C}_K$, we get a natural L-linear map

$$D_{\mathrm{dR},K}(V) \otimes_K L \longrightarrow D_{\mathrm{dR},L}(V).$$

It is evident that this map induces a morphism of filtered vector spaces over L where the filtrations on the source and the target are given as in Example 2.4.2. We then have

$$\mathrm{Fil}^n(D_{\mathrm{dR},K}(V)) = \mathrm{Fil}^n(D_{\mathrm{dR},L}(V))^{\mathrm{Gal}(L/K)} \qquad \text{for all } n \in \mathbb{Z},$$

thereby deducing the desired assertion by the Galois descent for vector spaces. □

Remark. Proposition 2.4.14 extends to any complete discrete-valued extension L of K inside $\mathbb{C}_K$, based on the "completed unramified descent argument" as explained in [**BC**, Proposition 6.3.8]. This fact has the following immediate consequences:

(1) Every potentially unramified p-adic representation is de Rham; indeed, we have already mentioned this in Example 2.4.2 since being $\mathbb{C}_K$-admissible is the same as being potentially unramified as noted in Example 1.1.4.
(2) For one-dimensional p-adic representations, being de Rham is the same as being Hodge-Tate by Proposition 1.1.13 and Lemma 2.4.3.

Example 2.4.15. Let $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^\times$ be a continuous character with finite image. Then there exists a finite extension L of K with absolute Galois group Γ_L such that $\mathbb{Q}_p(\eta)$ is trivial as a representation of Γ_L. Hence by Example 2.4.7 and Proposition 2.4.14 we find that $\mathbb{Q}_p(\eta)$ is de Rham with an isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) \otimes_K L \cong L[0],$$

and consequently obtain an identification

$$D_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) \cong K[0] \cong D_{\mathrm{dR}}(\mathbb{Q}_p).$$

In particular, we deduce that the functor D_{dR} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ with values in Fil_K is not full.

We close this section by introducing a very important conjecture, known as the *Fontaine-Mazur conjecture*, which predicts a criterion for the "geometricity" of global p-adic representations.

Conjecture 2.4.16 (Fontaine-Mazur [**FM95**]). *Fix a number field E, and denote by $\mathcal{O}_E$ the ring of integers in E. Let V be a finite dimensional representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ *over $\mathbb{Q}_p$ with the following properties:*

(i) For all but finitely many prime ideals $\mathfrak{p}$ of $\mathcal{O}_E$, the representation V is unramified at $\mathfrak{p}$ in the sense that the action of the inertia group at $\mathfrak{p}$ is trivial.
(ii) For all prime ideals of $\mathcal{O}_E$ lying over p, the restriction of V to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E_{\mathfrak{p}})$ *is de Rham.*

Then there exist a proper smooth variety X over E such that V appears as a subquotient of the étale cohomology $H^n_{ét}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(m))$ for some $m, n \in \mathbb{Z}$.

Remark. If V is one-dimensional, then Conjecture 2.4.16 follows essentially by the class field theory. For two-dimensional representations, Conjecture 2.4.16 has been verified in many cases by the work of Kisin and Emerton. However, Conjecture 2.4.16 remains wide open for higher dimensional representations.

The local version of Conjecture 2.4.16 is known to be false. More precisely, there exists a de Rham representation of Γ_K which does not arises as a subquotient of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)(m)$ for any proper smooth variety X over K and integers n, m.

3. Crystalline representations

In this section we define and study the crystalline period ring and crystalline representations. Our primary reference for this section is Brinon and Conrad's notes [**BC**, §9].

3.1. The crystalline period ring B_{cris}

Throughout this section, we write $W(k)$ for the ring of Witt vectors over k, and K_0 for its fraction field. Recall that we have fixed an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$ and set $\xi = [p^\flat] - p \in A_{\text{inf}}$.

Definition 3.1.1. We define the *integral crystalline period ring* by

$$A_{\text{cris}} := \left\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \in B_{\text{dR}}^+ : a_n \in A_{\text{inf}} \text{ with } \lim_{n\to\infty} a_n = 0 \right\},$$

and write $B_{\text{cris}}^+ := A_{\text{cris}}[1/p]$.

Remark. In the definition of A_{cris} above, it is vital to consider the refinement of the discrete valuation topology on B_{dR}^+ as described in Proposition 2.2.16. While the convergence of the infinite sum $\sum_{n\geq 0} a_n \frac{\xi^n}{n!}$ relies on the discrete valuation topology on B_{dR}^+, the limit of the coefficients a_n should be taken with respect to the p-adic topology on A_{inf}.

We warn the readers that the terminology given in Definition 3.1.1 is not standard at all. In fact, most authors do not give a separate name for the ring A_{cris}. Our choice of the terminology comes from the fact that A_{cris} plays the role of the crystalline period ring in the integral p-adic Hodge theory.

Proposition 3.1.2. *We have $t \in A_{\text{cris}}$ and $t^{p-1} \in pA_{\text{cris}}$.*

PROOF. By Lemma 2.2.18 we may write $[\varepsilon] - 1 = \xi c$ for some $c \in A_{\text{inf}}$. Then we obtain an expression

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)! c^n \cdot \frac{\xi^n}{n!}. \tag{3.1}$$

We thus find $t \in A_{\text{cris}}$ as we have $\lim_{n\to\infty} (n-1)!c^n = 0$ in A_{inf} relative to the p-adic topology.

It remains to show $t^{p-1} \in pA_{\text{cris}}$. Let us set

$$\check{t} := \sum_{n=1}^{p} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n}. \tag{3.2}$$

Since $(n-1)!$ is divisible by p for all $n > p$, we find $t - \check{t} \in pA_{\text{cris}}$ by (3.1). Hence it suffices to prove $\check{t}^{p-1} \in pA_{\text{cris}}$.

The terms for $n < p$ in (3.2) are all divisible by $[\varepsilon] - 1$ in A_{cris}, whereas the term for $n = p$ in (3.2) can be written as

$$(-1)^{p+1} \frac{([\varepsilon]-1)^p}{p} = (-1)^{p+1} \frac{([\varepsilon]-1)^{p-1}}{p} \cdot ([\varepsilon]-1).$$

In other words, we may write

$$\check{t} = ([\varepsilon]-1) \left(a + (-1)^{p+1} \frac{([\varepsilon]-1)^{p-1}}{p} \right)$$

for some $a \in A_{\text{cris}}$. It is therefore enough to show $([\varepsilon]-1)^{p-1} \in pA_{\text{cris}}$.

Since we have $([\varepsilon]-1)-[\varepsilon-1] \in pA_{\text{inf}} \subseteq pA_{\text{cris}}$, we only need to prove $[(\varepsilon-1)^{p-1}] \in pA_{\text{cris}}$. In addition, by Lemma 2.2.17 we have

$$\nu^\flat\left((\varepsilon-1)^{p-1}\right) = p = \nu^\flat\left((p^\flat)^p\right),$$

and consequently find that $[(\varepsilon-1)^{p-1}]$ is divisible by $[p^\flat]^p = (\xi+p)^p$. We thus deduce the desired assertion by observing that $\xi^p = p \cdot (p-1)! \cdot (\xi^p/p!)$ is divisible by p in A_{cris}. □

Remark. As a consequence, we find

$$\frac{t^p}{p!} = \frac{t^{p-1}}{p} \cdot \frac{t}{(p-1)!} \in A_{\text{cris}}.$$

In fact, it is not hard to prove that for every $a \in A_{\text{cris}}$ with $\theta^+_{\text{dR}}(a) = 0$ we have $a^n/n! \in A_{\text{cris}}$ for all $n \geq 1$.

Corollary 3.1.3. *We have an identification* $B^+_{\text{cris}}[1/t] = A_{\text{cris}}[1/t]$.

Proof. Proposition 3.1.2 implies that p is a unit in $A_{\text{cris}}[1/t]$, thereby yielding

$$B^+_{\text{cris}}[1/t] = A_{\text{cris}}[1/p, 1/t] = A_{\text{cris}}[1/t]$$

as desired. □

Definition 3.1.4. We define the *crystalline period ring* by

$$B_{\text{cris}} := B^+_{\text{cris}}[1/t] = A_{\text{cris}}[1/t].$$

Remark. Let us briefly explain Fontaine's insight behind the construction of B_{cris}. The main motivation for constructing the crystalline period ring B_{cris} is to obtain the Grothendieck mysterious functor as described in Chapter I, Conjecture 1.2.3. Recall that, for a proper smooth variety X over K with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$, the crystalline cohomology $H^n_{\text{cris}}(\mathcal{X}_k, W(k))$ admits a natural Frobenius action and refines the de Rham cohomology $H^n_{\text{dR}}(X/K)$ via a canonical isomorphism

$$H^n_{\text{cris}}(\mathcal{X}_k, W(k))[1/p] \otimes_{K_0} K \cong H^n_{\text{dR}}(X/K).$$

In addition, since A_{inf} is by construction the ring of Witt vectors over a perfect $\mathbb{F}_p$-algebra $\mathcal{O}_F$, it admits the Frobenius automorphism φ_{inf} as noted in Chapter II, Example 2.3.2. Fontaine sought to construct B_{cris} as a sufficiently large subring of B_{dR} on which φ_{inf} naturally extends. For B_{dR} there is no natural extension of φ_{inf} since $\ker(\theta[1/p])$ is not stable under φ_{inf}. Fontaine's key observation is that by adjoining to A_{inf} the elements of the form $\xi^n/n!$ for $n \geq 1$ we obtain a subring of $A_{\text{inf}}[1/p]$ such that the image of $\ker(\theta[1/p])$ is stable under φ_{inf}. This observation led Fontaine to consider the ring A_{cris} defined in Definition 3.1.1. The only issue with A_{cris} is that it is not $(\mathbb{Q}_p, \Gamma_K)$-regular, which turns out to be resolved by considering the ring $B_{\text{cris}} = A_{\text{cris}}[1/t]$.

Proposition 3.1.5. *The ring* B_{cris} *is naturally a filtered subalgebra of* B_{dR} *over* K_0 *which is stable under the action of* Γ_K.

Proof. By construction we have

$$A_{\text{inf}}[1/p] \subseteq A_{\text{cris}}[1/p] = B^+_{\text{cris}} \subseteq B_{\text{cris}} \subseteq B_{\text{dR}}.$$

In addition, the proof of Proposition 2.2.15 yields a unique homomorphism $K \longrightarrow B_{\text{dR}}$ which extends a natural homomorphism $K_0 \longrightarrow A_{\text{inf}}[1/p]$. Hence by Example 2.3.2 we naturally identify B_{cris} as a filtered subalgebra of B_{dR} over K_0 with $\text{Fil}^n(B_{\text{cris}}) := B_{\text{cris}} \cap t^n B^+_{\text{dR}}$.

It remains to show that $B_{\text{cris}} = A_{\text{cris}}[1/t]$ is stable under the action of Γ_K. Since Γ_K acts on t by the cyclotomic character as noted in Theorem 2.2.21, we only need to show that A_{cris}

is stable under the action of Γ_K. Consider an arbitrary element $\gamma \in \Gamma_K$ and an arbitrary sequence (a_n) in A_{inf} with $\lim\limits_{n\to\infty} a_n = 0$. Since $\ker(\theta)$ is stable under the Γ_K-action as noted in Theorem 2.2.21, we may write $\gamma(\xi) = c_\gamma \xi$ for some $c_\gamma \in A_{\text{inf}}$ by Proposition 2.2.6. We then have $\lim\limits_{n\to\infty} \gamma(a_n)c_\gamma^n = 0$ as the Γ_K-action on A_{inf} is evidently continuous with respect to the p-adic topology. Hence we find

$$\gamma\left(\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}\right) = \sum_{n=0}^{\infty} \gamma(a_n)c_\gamma^n \frac{\xi^n}{n!} \in A_{\text{cris}}$$

as desired. □

Remark. We provide a functorial perspective for the Γ_K-actions on B_{cris} and B_{dR} which can be useful in many occasions. Since the definitions of B_{cris} and B_{dR} only depend on the valued field $\mathbb{C}_K$, we may regard B_{cris} and B_{dR} as functors which associate topological rings to each complete and algebraically closed valued field. Then by functoriality the action of Γ_K on $\mathbb{C}_K$ induces the actions of Γ_K on B_{cris} and B_{dR}. In particular, since B_{cris} is a subfunctor of B_{dR} we deduce that the Γ_K-action on B_{cris} is given by the restriction of the Γ_K-action on B_{dR} as asserted in Proposition 3.1.5.

We also warn that $\text{Fil}^0(B_{\text{cris}}) = B_{\text{cris}} \cap B_{\text{dR}}^+$ is not equal to B_{cris}^+. For example, the element

$$\alpha = \frac{[\varepsilon^{1/p^2}] - 1}{[\varepsilon^{1/p}] - 1}$$

lies in $B_{\text{cris}} \cap B_{\text{dR}}^+$ but not in B_{cris}^+.

In order to study the Γ_K-action on B_{cris} we invoke the following crucial (and surprisingly technical) result without proof.

Proposition 3.1.6. *The natural Γ_K-equivariant map $B_{\text{cris}} \otimes_{K_0} K \longrightarrow B_{\text{dR}}$ is injective.*

Remark. The original proof by Fontaine in [**Fon94**] is incorrect. A complete proof involving the semistable period ring can be found in Fontaine and Ouyang's notes [**FO**, Theorem 6.14]. Note however that the assertion is obvious if we have $K = K_0$, which amounts to the condition that K is unramified over $\mathbb{Q}_p$.

Proposition 3.1.7. *There exists a natural isomorphism of graded K-algebras*

$$\text{gr}(B_{\text{cris}} \otimes_{K_0} K) \cong \text{gr}(B_{\text{dR}}) \cong B_{\text{HT}}.$$

Proof. We only need to establish the first identification as the second identification immediately follows from Theorem 2.2.21 as noted in Example 2.3.2. By Proposition 3.1.6 the natural map $B_{\text{cris}} \otimes_{K_0} K \longrightarrow B_{\text{dR}}$ induces an injective morphism of graded K-algebras

$$\text{gr}(B_{\text{cris}} \otimes_{K_0} K) \hookrightarrow \text{gr}(B_{\text{dR}}). \tag{3.3}$$

In particular, we have an injective map

$$\text{gr}^0(B_{\text{cris}} \otimes_{K_0} K) \hookrightarrow \text{gr}^0(B_{\text{dR}}) \cong \mathbb{C}_K$$

where the isomorphism is induced by θ_{dR}^+. Moreover, this map is surjective since the image of $B_{\text{cris}} \otimes_{K_0} K$ in B_{dR} contains $A_{\text{inf}}[1/p]$ and consequently maps onto $\mathbb{C}_K$ by θ_{dR}^+. Therefore we obtain an isomorphism

$$\text{gr}^0(B_{\text{cris}} \otimes_{K_0} K) \cong \text{gr}^0(B_{\text{dR}}) \cong \mathbb{C}_K.$$

This implies that each $\text{gr}^n(B_{\text{cris}} \otimes_{K_0} K)$ is a vector space over $\mathbb{C}_K$. Moreover, each $\text{gr}^n(B_{\text{cris}} \otimes_{K_0} K)$ contains a nonzero element given by $t^n \otimes 1$. Hence the injective map (3.3) must be an isomorphism since each $\text{gr}^n(B_{\text{dR}})$ has dimension 1 over $\mathbb{C}_K$. □

Theorem 3.1.8 (Fontaine [**Fon94**]). *The ring B_{cris} is $(\mathbb{Q}_p, \Gamma_K)$-regular with $B_{\text{cris}}^{\Gamma_K} \cong K_0$.*

PROOF. Let C_{cris} denote the fraction field of B_{cris}. Proposition 3.1.5 implies that C_{cris} is a subfield of B_{dR} which is stable under the action of Γ_K. Hence we have $K_0 \subseteq B_{\text{cris}}^{\Gamma_K} \subseteq C_{\text{cris}}^{\Gamma_K}$. Then Proposition 3.1.6 and Theorem 2.2.21 together yield injective maps

$$B_{\text{cris}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\text{dR}}^{\Gamma_K} \cong K \qquad \text{and} \qquad C_{\text{cris}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\text{dR}}^{\Gamma_K} \cong K,$$

thereby implying $K_0 = B_{\text{cris}}^{\Gamma_K} = C_{\text{cris}}^{\Gamma_K}$.

It remains to check the condition (ii) in Definition 1.1.1. Consider an arbitrary nonzero element $b \in B_{\text{cris}}$ on which Γ_K acts via a character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$. We wish to show that b is a unit in B_{cris}.

By Proposition 2.2.19 we may write $b = t^i b'$ for some $b' \in (B_{\text{dR}}^+)^\times$ and $i \in \mathbb{Z}$. Since t is a unit in B_{cris} by construction, the element b is a unit in B_{cris} if and only if b' is a unit in B_{cris}. Moreover, Theorem 2.2.21 implies that Γ_K acts on $b' = b \cdot t^{-i}$ via the character $\eta\chi^{-i}$. Hence we may replace b by b' to assume that b is a unit in B_{dR}^+.

Since θ_{dR}^+ is Γ_K-equivariant as noted in Theorem 2.2.21, the action of Γ_K on $\theta_{\text{dR}}^+(b) \in \mathbb{C}_K$ is given by the character η. Then by the continuity of the Γ_K-action on $\mathbb{C}_K$ we find that η is continuous. Therefore we may consider η as a character with values in $\mathbb{Z}_p^\times$. Moreover, we have $\theta_{\text{dR}}^+(b) \neq 0$ as b is assumed to be a unit in B_{dR}^+. Hence Theorem 1.1.8 implies that $\eta^{-1}(I_K)$ is finite.

Let us denote by K^{un} the maximal unramified extension of K in $\overline{K}$, and by $\widehat{K^{\text{un}}}$ the p-adic completion of K^{un}. By definition $\widehat{K^{\text{un}}}$ is a p-adic field with I_K as the absolute Galois group. Therefore by our discussion in the preceding paragraph there exists a finite extension L of $\widehat{K^{\text{un}}}$ with the absolute Galois group Γ_L such that η^{-1} becomes trivial on $\Gamma_L \subseteq I_K$. Since Γ_K acts on $\theta_{\text{dR}}^+(b)$ via η, we find $\theta_{\text{dR}}^+(b) \in \mathbb{C}_K^{\Gamma_L} = \mathbb{C}_L^{\Gamma_L} = L$ by Theorem 3.1.12 in Chapter II.

Let us write $W(\overline{k})$ for the ring of Witt vectors over $\overline{k}$, and $\widehat{K_0^{\text{un}}}$ for the fraction field of $W(\overline{k})$. Proposition 2.2.15 yields a commutative diagram

$$\begin{array}{ccc} \widehat{K_0^{\text{un}}} & \longrightarrow & A_{\text{inf}}[1/p] \\ \big\downarrow & & \big\downarrow \\ L & \longrightarrow & B_{\text{dR}}^+ \\ & \searrow & \big\downarrow {\scriptstyle \theta_{\text{dR}}^+} \\ & & \mathbb{C}_K \end{array} \tag{3.4}$$

where all maps are Γ_K-equivariant. Moreover, both horizontal maps are injective as $\widehat{K_0^{\text{un}}}$ and L are fields. We henceforth identify $\widehat{K_0^{\text{un}}}$ and L with their images in B_{dR}. Then we have

$$\widehat{K_0^{\text{un}}} \subseteq A_{\text{inf}}[1/p] \subseteq B_{\text{cris}}. \tag{3.5}$$

We assert that b lies in (the image of) L. Let us write $\hat{b} := \theta_{\text{dR}}^+(b)$. If suffices to show $b = \hat{b}$. Suppose for contradiction that b and $\hat{b}$ are distinct. Since we have $\theta_{\text{dR}}^+(\hat{b}) = \hat{b} = \theta_{\text{dR}}^+(b)$ by the commutativity of the diagram (3.4), we may write $b - \hat{b} = t^j u$ for some $j > 0$ and $u \in (B_{\text{dR}}^+)^\times$. Moreover, we find

$$\gamma(b - \hat{b}) = \gamma(b) - \gamma(\hat{b}) = \eta(\gamma)(b - \hat{b}) \qquad \text{for every } \gamma \in \Gamma_K.$$

Then under the Γ_K-equivariant isomorphism

$$t^j B_{\mathrm{dR}}^+/t^{j+1}B_{\mathrm{dR}}^+ \cong \mathbb{C}_K(j)$$

given by Theorem 2.2.21, the element $b - \hat{b} \in t^j B_{\mathrm{dR}}^+$ yields a nonzero element in $\mathbb{C}_K(j)$ on which Γ_K acts via the character η. Therefore Theorem 1.1.8 implies that $(\chi^j\eta^{-1})(I_K)$ is finite. Since $\eta^{-1}(I_K)$ is also finite as noted above, we deduce that $\chi^j(I_K)$ is finite as well, thereby obtaining a desired contradiction by Lemma 1.1.7.

Let us now regard b as an element in L. Proposition 2.2.15 implies that L is a finite extension of $\widehat{K_0^{\mathrm{un}}}$. Hence we can choose a minimal polynomial equation

$$b^d + a_1 b^{d-1} + \cdots + a_{d-1}b + a_d = 0 \qquad \text{with } a_n \in \widehat{K_0^{\mathrm{un}}}.$$

Since the minimality of the equation implies $a_d \neq 0$, we obtain an expression

$$b^{-1} = -a_d^{-1}(b^{d-1} + a_1 b^{d-2} + \cdots + a_{d-1}).$$

We then find $b^{-1} \in B_{\mathrm{cris}}$ by (3.5), thereby completing the proof. □

Our final goal in this subsection is to construct the Frobenius endomorphism on B_{cris}. To this end we state another technical result without proof.

Proposition 3.1.9. *Let A_{cris}^0 be the A_{inf}-subalgebra in $A_{\mathrm{inf}}[1/p]$ generated by the elements of the form $\xi^n/n!$ with $n \geq 0$.*

(1) The ring A_{cris} is naturally identified with the p-adic completion of A_{cris}^0.
(2) The action of Γ_K on A_{cris} is continuous.

Remark. In fact, Fontaine originally defined the ring A_{cris} as the p-adic completion of A_{cris}^0, and obtained an identification with our definition of A_{cris}. The proof requires yet another description of the ring A_{cris} as a p-adically completed tensor product. The readers can find a sketch of the proof in [**BC**, Proposition 9.1.1 and Proposition 9.1.2].

Lemma 3.1.10. *The Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B_{cris}^+.*

Proof. The Frobenius automorphism of A_{inf} uniquely extends to an automorphism on $A_{\mathrm{inf}}[1/p]$, which we denote by φ_{inf}. By construction we have

$$\varphi_{\mathrm{inf}}(\xi) = [(p^\flat)^p] - p = [p^\flat]^p - p = (\xi + p)^p - p. \tag{3.6}$$

Hence we may write $\varphi_{\mathrm{inf}}(\xi) = \xi^p + pc$ for some $c \in A_{\mathrm{inf}}$.

Let us define A_{cris}^0 as in Proposition 3.1.9. Then we have

$$\varphi_{\mathrm{inf}}(\xi) = p \cdot (c + (p-1)! \cdot (\xi^p/p!)),$$

and consequently find

$$\varphi_{\mathrm{inf}}(\xi^n/n!) = (p^n/n!) \cdot (c + (p-1)! \cdot (\xi^p/p!))^n \in A_{\mathrm{cris}}^0 \qquad \text{for all } n \geq 1$$

by observing that $p^n/n!$ is an element of $\mathbb{Z}_p$. Hence A_{cris}^0 is stable under φ_{inf}. Moreover, by construction φ_{inf} is Γ_K-equivariant and continuous on $A_{\mathrm{inf}}[1/p]$ with respect to the p-adic topology. We thus deduce by Proposition 3.1.9 that the endomorphism φ_{inf} on A_{cris}^0 uniquely extends to a continuous Γ_K-equivariant endomorphism φ^+ on $B_{\mathrm{cris}}^+ = A_{\mathrm{cris}}[1/p]$. □

Remark. The identity (3.6) shows that $\varphi_{\mathrm{inf}}(\xi)$ is not divisible by ξ, which implies that $\ker(\theta)$ is not stable under φ_{inf}. Hence the endomorphism φ^+ on B_{cris}^+ (or the Frobenius endomorphism on B_{cris} that we are about to construct) is not compatible with the filtration on B_{dR}.

Proposition 3.1.11. *The Frobenius automorphism of A_{inf} naturally extends to a Γ_K-equivariant endomorphism φ on B_{cris} with $\varphi(t) = pt$.*

Proof. As noted in Lemma 3.1.10, the Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B^+_{cris}. In addition, the proof of Proposition 3.1.2 shows that the power series expression

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n}$$

converges with respect to the p-adic topology in A_{cris}. Hence we use Lemma 2.2.20 and the continuity of φ^+ on A_{cris} to find

$$\varphi^+(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)^n}{n} = \log(\varepsilon^p) = p \log(\varepsilon) = pt.$$

Since Γ_K acts on t via χ, it follows that φ^+ uniquely extends to a Γ_K-equivariant endomorphism φ on $B_{\text{cris}} = B^+_{\text{cris}}[1/t]$. □

Remark. The endomorphism φ is not continuous on B_{cris}, even though it is a unique extension of the continuous endomorphism φ^+ on B^+_{cris}. The issue is that, as pointed out by Colmez in [**Col98**], the natural topology on B^+_{cris} induced by the p-adic topology on A_{cris} does not agree with the subspace topology inherited from B_{cris}.

Definition 3.1.12. We refer to the endomorphism φ in Proposition 3.1.11 as the *Frobenius endomorphism* of B_{cris}. We also write

$$B_e := \{\, b \in B_{\text{cris}} : \varphi(b) = b \,\}$$

for the ring of Frobenius-invariant elements in B_{cris}.

Remark. In Chapter IV, we will use the Fargues-Fontaine curve to prove a surprising fact that B_e is a principal ideal domain.

We close this subsection by stating two fundamental results about φ without proof.

Theorem 3.1.13. *The Frobenius endomorphism φ of B_{cris} is injective.*

Theorem 3.1.14. *The natural sequence*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\text{dR}}/B^+_{\text{dR}} \longrightarrow 0$$

is exact.

Remark. We will prove both Theorem 3.1.13 and Theorem 3.1.14 in Chapter IV using the Fargues-Fontaine curve. There will be no circular reasoning; the construction of the Fargues-Fontaine curve does not rely on anything that we haven't discussed so far in this section. The readers can also find a proof of Theorem 3.1.14 in [**FO**, Theorem 6.26]. We also remark that, as mentioned in [**BC**, Theorem 9.1.8], there was no published proof of Theorem 3.1.13 prior to the work of Fargues-Fontaine [**FF18**].

Definition 3.1.15. We refer to the exact sequence in Theorem 3.1.14 as the *fundamental exact sequence of p-adic Hodge theory.*

3.2. Properties of crystalline representations

Definition 3.2.1. We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is *crystalline* if it is B_{cris}-admissible. We write $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\Gamma_K) := \mathrm{Rep}^{B_{\mathrm{cris}}}_{\mathbb{Q}_p}(\Gamma_K)$ for the category of crystalline p-adic Γ_K-representations. In addition, we write D_{cris} the functors $D_{B_{\mathrm{cris}}}$.

Example 3.2.2. We record some essential examples of crystalline representations.

(1) Every Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is crystalline; indeed, the inequality
$$\dim_K D_{\mathrm{cris}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$
given by Theorem 1.2.1 is an equality, as $D_{\mathrm{cris}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$ by Theorem 2.2.21.

(2) For every proper smooth variety X over K with with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline by a theorem of Faltings as discussed in Chapter I, Theorem 1.2.4; moreover, there exists a canonical isomorphism
$$D_{\mathrm{cris}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H_{\mathrm{cris}}(\mathcal{X}_k/K_0) = H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$$
where $H_{\mathrm{cris}}(\mathcal{X}_k/W(k))$ denotes the crystalline cohomology of $\mathcal{X}_k$.

(3) For every p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ is crystalline as proved by Fontaine; indeed, there exists a natural identification
$$D_{\mathrm{cris}}(V_p(G)) \cong \mathbb{D}(\overline{G})[1/p]$$
where $\mathbb{D}(\overline{G})$ denotes the Dieudonné module associated to $\overline{G} := G \times_{\mathcal{O}_K} k$ as described in Chapter II, Theorem 2.3.6.

We aim to promote D_{cris} to a functor that incorporates both the Frobenius endomorphism and the filtration on B_{cris}. Let us denote by σ the Frobenius automorphism of K_0 as defined in Chapter II, Definition 2.3.3. The readers may wish to review the definition and basic properties of isocrystals as discussed in Chapter II, Definition 2.3.3 and Lemma 2.3.4.

Definition 3.2.3. A *filtered isocrystal* over K is an isocrystal N over K_0 together with a collection of K-spaces $\{ \mathrm{Fil}^n(N_K) \}$ which yields a structure of a filtered vector space over K on $N_K := N \otimes_{K_0} K$. We denote by MF^{φ}_K the category of filtered isocrystals over K with the natural notions of morphisms, tensor products, and duals inherited from the corresponding notions for Fil_K and the category of isocrystals over K_0.

Remark. Many authors use an alternative terminology *filtered φ-modules.*

Example 3.2.4. Let X be a proper smooth variety over K with a proper smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$. The crystalline cohomology $H_{\mathrm{cris}}(\mathcal{X}_k/K_0) = H^n_{\mathrm{cris}}(\mathcal{X}_k/W(k))[1/p]$ is naturally a filtered isocrystal over K with the Frobenius automorphism $\varphi^*_{\mathcal{X}_k}$ induced by the relative Frobenius of $\mathcal{X}_K$ and the filtration on $H^n_{\mathrm{cris}}(\mathcal{X}_k/K_0) \otimes_{K_0} K$ given by the Hodge filtration on the de Rham cohomology $H^n_{\mathrm{dR}}(X/K)$ via the canonical comparison isomorphism
$$H^n_{\mathrm{cris}}(\mathcal{X}_k/K_0) \otimes_{K_0} K \cong H^n_{\mathrm{dR}}(X/K).$$

Lemma 3.2.5. *The automorphism σ on K_0 extends to the endomorphism φ on B_{cris}.*

Proof. By the proof of Proposition 2.2.15, the natural injective map $K_0 \hookrightarrow A_{\mathrm{inf}}[1/p]$ is a unique lift of the natural map $k \longrightarrow \mathcal{O}_F$. Hence σ extends to φ_{inf} on $A_{\mathrm{inf}}[1/p]$ by definition, and consequently extends to φ by Proposition 3.1.12. $\square$

Lemma 3.2.6. *Let N be a finite dimensional vector space over K_0. Every injective σ-semilinear additive map $f : N \longrightarrow N$ is bijective.*

Proof. The additivity of f implies that $f(N)$ is closed under addition. Moreover, for all $c \in K_0$ and $n \in N$ we have

$$cf(n) = \sigma(\sigma^{-1}(c))f(n) = f(\sigma^{-1}(c)n) \in f(N).$$

Therefore $f(N)$ is a subspace of N over K_0. We wish to show $f(N) = N$. Let us choose a basis (n_i) for N over K_0. It suffices to prove that the vectors $f(n_i)$ are linearly independent over K_0. Assume for contradiction that there exists a nontrivial a relation $\sum c_i f(n_i) = 0$ with $c_i \in K_0$. Then we find $f\left(\sum \sigma(c_i)n_i\right) = 0$ by the σ-semilinearity of f, and consequently obtain a relation $\sum \sigma(c_i)n_i = 0$ by the injectivity of f. Hence we have a nontrivial relation among the vectors n_i as σ is an automorphism on K_0, thereby obtaining contradiction as desired. □

Proposition 3.2.7. *Let V be a p-adic representation of Γ_K. Then $D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ is naturally a filtered isocrystal over K with the Frobenius automorphism $1 \otimes \varphi$ and the filtration on $D_{\mathrm{cris}}(V)_K = D_{\mathrm{cris}}(V) \otimes_{K_0} K$ given by*

$$\mathrm{Fil}^n(D_{\mathrm{cris}}(V)_K) := (V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^n(B_{\mathrm{cris}} \otimes_{K_0} K))^{\Gamma_K}.$$

Proof. Since Γ_K acts trivially on K, we have a natural identification

$$D_{\mathrm{cris}}(V)_K = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K))^{\Gamma_K}.$$

Then Proposition 3.1.6 implies that $D_{\mathrm{cris}}(V)_K$ is a filtered vector space over K with the filtration $\mathrm{Fil}^n(D_{\mathrm{cris}}(V)_K)$ as defined above. Therefore it remains to verify that the map $1 \otimes \varphi$ is σ-semilinear and bijective on $D_{\mathrm{cris}}(V)$. For arbitrary $v \in V, b \in B_{\mathrm{cris}}$, and $c \in K_0$ we have

$$(1 \otimes \varphi)(c(v \otimes b)) = (1 \otimes \varphi)(v \otimes bc) = v \otimes \varphi(b)\varphi(c) = \varphi(c) \cdot (1 \otimes \varphi)(v \otimes b).$$

Hence by Lemma 3.2.5 we find that the additive map $1 \otimes \varphi$ is σ-semilinear. Moreover, the map $1 \otimes \varphi$ is injective on $D_{\mathrm{cris}}(K)$ by Theorem 3.1.13 and the left exactness of the functor D_{cris}. Thus we deduce the desired assertion by Lemma 3.2.6. □

Proposition 3.2.8. *Let V be a crystalline representation of Γ_K. Then V is de Rham with a natural isomorphism of filtered vector spaces*

$$D_{\mathrm{cris}}(V)_K = D_{\mathrm{cris}}(V) \otimes_{K_0} K \cong D_{\mathrm{dR}}(V).$$

Proof. Proposition 3.1.5 and Proposition 3.1.6 together imply that the natural map $B_{\mathrm{cris}} \otimes_{K_0} K \longrightarrow B_{\mathrm{dR}}$ identifies $B_{\mathrm{cris}} \otimes_{K_0} K$ as a filtered subspace of B_{dR} over K; in other words, we have an identification

$$\mathrm{Fil}^n(B_{\mathrm{cris}} \otimes_{K_0} K) = (B_{\mathrm{cris}} \otimes_{K_0} K) \cap \mathrm{Fil}^n(B_{\mathrm{dR}}) \qquad \text{for every } n \in \mathbb{Z}.$$

Therefore Proposition 3.2.7 yields a natural injective morphism of filtered vector spaces

$$D_{\mathrm{cris}}(V)_K = (V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K))^{\Gamma_K} \hookrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K} = D_{\mathrm{dR}}(V)$$

with an identification

$$\mathrm{Fil}^n(D_{\mathrm{cris}}(V) \otimes_{K_0} K) = (D_{\mathrm{cris}}(V) \otimes_{K_0} K) \cap \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) \qquad \text{for every } n \in \mathbb{Z}.$$

We then find

$$\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_K D_{\mathrm{cris}}(V)_K \leq \dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from Theorem 1.2.1. Since V is crystalline, both inequalities should be in fact equalities, thereby yielding the desired assertion. □

Example 3.2.9. Let $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ be a nontrivial continuous character which factors through $\mathrm{Gal}(L/K)$ for some totally ramified finite extension L of K. Then $\mathbb{Q}_p(\eta)$ is de Rham by Proposition 2.4.14. We assert that $\mathbb{Q}_p(\eta)$ is not crystalline. Let us write Γ_L for the absolute Galois group of L. Since L is totally ramified over K, we have $B_{\mathrm{cris}}^{\Gamma_L} \cong K_0$ by Theorem 3.1.8 and the fact that the construction of B_{cris} depends only on $\mathbb{C}_K$. Moreover, we have $\mathbb{Q}_p(\eta)^{\Gamma_L} = \mathbb{Q}_p(\eta)$ and $\mathbb{Q}_p(\eta)^{\mathrm{Gal}(L/K)} = 0$ by construction. Hence we find an identification

$$\begin{aligned} D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)) &= (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} = \left((\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_L}\right)^{\mathrm{Gal}(L/K)} \\ &= \left(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^{\Gamma_L}\right)^{\mathrm{Gal}(L/K)} \cong (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} K_0)^{\mathrm{Gal}(L/K)} \\ &= \mathbb{Q}_p(\eta)^{\mathrm{Gal}(L/K)} \otimes_{\mathbb{Q}_p} K_0 = 0, \end{aligned}$$

thereby deducing the desired assertion.

We now adapt the argument in §2.4 to verify that the general formalism discussed in §1 extends to the category of crystalline representations with the enhanced functor D_{cris} that takes values in MF_K^{φ}.

Proposition 3.2.10. *Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ induces a natural Γ_K-equivariant isomorphism*

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

which is compatible with the natural Frobenius endomorphisms on both sides and induces a K-linear isomorphism of filtered vector spaces

$$D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

Proof. Since V is crystalline, Theorem 1.2.1 implies that the natural map

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} B_{\mathrm{cris}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

is a Γ_K-equivariant B_{cris}-linear isomorphism. Moreover, this map is visibly compatible with the natural Frobenius endomorphisms on $D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \otimes_{K_0} B_{\mathrm{cris}}$ and $V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$ respectively given by $1 \otimes \varphi \otimes \varphi$ and $1 \otimes \varphi$. Let us now consider the induced K-linear bijective map

$$(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \longrightarrow V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

It is straightforward to check that this map is a morphism of filtered vector spaces. Therefore by Proposition 2.3.8 it suffices to show that the induced map

$$\mathrm{gr}\,(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)) \longrightarrow \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \tag{3.7}$$

is an isomorphism. As V is crystalline, it is also Hodge-Tate with the natural isomorphism of graded vector spaces

$$\mathrm{gr}(D_{\mathrm{cris}}(V)_K) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V)$$

by Proposition 3.2.8 and Proposition 2.4.4. Hence Proposition 2.3.10 and Proposition 3.1.7 together yield identifications

$$\begin{aligned} \mathrm{gr}\,(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)) &\cong \mathrm{gr}(D_{\mathrm{cris}}(V)_K) \otimes_K \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}}, \\ \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}. \end{aligned}$$

We thus identify the map (3.7) with the natural map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1, thereby deducing the desired assertion by the fact that V is Hodge-Tate. □

Proposition 3.2.11. *The functor D_{cris} with values in MF_K^{φ} is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$.*

Proof. Let Vec_{K_0} denote the category of finite dimensional vector spaces over K_0. The faithfulness of D_{cris} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ follows immediately from Proposition 1.2.2 since the forgetful functor $\mathrm{MF}_K^{\varphi} \longrightarrow \mathrm{Vec}_{K_0}$ is faithful. Hence it remains to verify the exactness of D_{cris} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$. Consider an arbitrary exact sequence of crystalline representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

We wish to show that the sequence

$$0 \longrightarrow D_{\mathrm{cris}}(U) \longrightarrow D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(W) \longrightarrow 0 \tag{3.8}$$

is exact in MF_K^{φ}. This sequence is exact in Vec_{K_0} by Proposition 1.2.2, and thus is also exact in the category of isocrystals over K_0. Moreover, Proposition 3.2.8 and Proposition 2.4.9 together imply that we can identify the induced sequence of filtered vector spaces

$$0 \longrightarrow D_{\mathrm{cris}}(U)_K \longrightarrow D_{\mathrm{cris}}(V)_K \longrightarrow D_{\mathrm{cris}}(W)_K \longrightarrow 0$$

with the exact sequence of filtered vector spaces

$$0 \longrightarrow D_{\mathrm{dR}}(U) \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(W) \longrightarrow 0$$

induced by (3.2). We thus deduce that the sequence (3.8) is exact in MF_K^{φ} as desired. □

Corollary 3.2.12. *Let V be a crystalline representation. Every subquotient W of V is a crystalline representation with $D_{\mathrm{cris}}(W)$ naturally identified as a subquotient of $D_{\mathrm{dR}}(V)$.*

Proof. This is an immediate consequence of Proposition 1.2.3 and Proposition 3.2.11. □

Proposition 3.2.13. *Given any $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$, we have $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ with a natural isomorphism of filtered isocrystals*

$$D_{\mathrm{cris}}(V) \otimes_{K_0} D_{\mathrm{cris}}(W) \cong D_{\mathrm{cris}}(V \otimes_{\mathbb{Q}_p} W). \tag{3.9}$$

Proof. By Proposition 1.2.4 we find $V \otimes_{\mathbb{Q}_p} W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ and obtain the desired isomorphism (3.9) as a map of vector spaces. Moreover, since the construction of the map (3.9) rests on the multiplicative structure of B_{cris} as shown in the proof of Proposition 1.2.4, it is straightforward to verify that the map (3.9) is a morphism of isocrystals over K_0. In addition, Proposition 3.2.8 implies that we can identify the induced bijective K-linear map

$$D_{\mathrm{cris}}(V)_K \otimes_K D_{\mathrm{cris}}(W)_K \longrightarrow D_{\mathrm{cris}}(V \otimes_{\mathbb{Q}_p} W)_K.$$

with the natural isomorphism of filtered vector spaces

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W)_K \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)$$

given by Proposition 2.4.11. Therefore we deduce that the map (3.9) is an isomorphism in MF_K^{φ} as desired. □

Proposition 3.2.14. *For every crystalline representation V, we have $\wedge^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ and $\mathrm{Sym}^n(V) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ with natural isomorphisms of filtered isocrystals*

$$\wedge^n(D_{\mathrm{cris}}(V)) \cong D_{\mathrm{cris}}(\wedge^n(V)) \qquad \text{and} \qquad \mathrm{Sym}^n(D_{\mathrm{cris}}(V)) \cong D_{\mathrm{cris}}(\mathrm{Sym}^n(V)).$$

Proof. Proposition 1.2.5 implies that both $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are crystalline for every $n \geq 1$. In addition, Proposition 1.2.5 yields the desired isomorphisms as maps of vector spaces. Then Corollary 3.2.12 and Proposition 3.2.13 together imply that these maps are isomorphisms in MF_K^{φ}. □

Proposition 3.2.15. *For every crystalline representation V, the dual representation $V^\vee$ is crystalline with a natural perfect pairing of filtered isocrystals*

$$D_{\mathrm{cris}}(V) \otimes_{K_0} D_{\mathrm{cris}}(V^\vee) \cong D_{\mathrm{cris}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\mathrm{cris}}(\mathbb{Q}_p).$$

PROOF. By Proposition 1.2.6 we find $V^\vee \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ and obtain the desired perfect pairing as a map of vector spaces. Moreover, Proposition 3.2.13 implies that this pairing is a morphism in MF_K^φ. We thus obtain a bijective morphism of filtered isocrystals

$$D_{\mathrm{cris}}(V)^\vee \longrightarrow D_{\mathrm{cris}}(V^\vee). \tag{3.10}$$

Furthermore, by Proposition 3.2.8 we identify the induced morphism of filtered vector spaces

$$D_{\mathrm{cris}}(V)_K^\vee \longrightarrow D_{\mathrm{cris}}(V^\vee)_K$$

with the natural isomorphism $D_{\mathrm{dR}}(V) \cong D_{\mathrm{dR}}(V^\vee)$ in Fil_K given by Proposition 2.4.13. Hence we deduce that the map (3.10) is an isomorphism in MF_K^φ, thereby completing the proof. □

Finally, we discuss some additional key properties of crystalline representations and the functor D_{cris} which resolve the main defects of de Rham representations and the functor D_{dR}.

Definition 3.2.16. Let M be a module over a ring R with an additive endomorphism f. For every $r \in R$, we refer to the subgroup

$$M^{f=r} := \{\, m \in M : f(m) = rm \,\}$$

as the *eigenspace of f* with eigenvalue r.

Lemma 3.2.17. *We have an identification*

$$B_{\mathrm{cris}}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K) = B_{\mathrm{cris}}^{\varphi=1} \cap B_{\mathrm{dR}}^+ = \mathbb{Q}_p.$$

PROOF. By Proposition 3.1.6 and Theorem 3.1.14 we find

$$B_{\mathrm{cris}}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K) \subseteq B_{\mathrm{cris}}^{\varphi=1} \cap \mathrm{Fil}^0(B_{\mathrm{dR}}) = B_{\mathrm{cris}}^{\varphi=1} \cap B_{\mathrm{dR}}^+ = \mathbb{Q}_p,$$

and thus obtain the desired identification as both $B_{\mathrm{cris}}^{\varphi=1}$ and $\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K)$ contain $\mathbb{Q}_p$. □

Proposition 3.2.18. *Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ admits canonical isomorphisms*

$$\begin{aligned} V &\cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \\ &\cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}}\right). \end{aligned}$$

PROOF. Proposition 3.2.10 yields a natural Γ_K-equivariant isomorphism

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

which is compatible with the natural Frobenius endomorphisms on both sides and induces an isomorphism of filtered vector spaces

$$D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

In addition, there exists a canonical isomorphism of filtered vector spaces

$$D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}} \cong D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

given by Proposition 3.2.8 and Proposition 2.4.8. Therefore we have identifications

$$\begin{gathered} (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^{\varphi=1}, \\ \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \cong V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K), \\ \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}}\right) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+. \end{gathered}$$

The desired assertion now follows by Lemma 3.2.17. □

Theorem 3.2.19 (Fontaine [**Fon94**]). *The functor D_{cris} with values in MF_K^{φ} is exact and fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$.*

Proof. By Proposition 3.2.11 we only need to establish the fullness of D_{cris} on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$. Let V and W be arbitrary crystalline representations. Consider an arbitrary morphism $f : D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(W)$ in MF_K^{φ}. Then f gives rise to a Γ_K-equivariant map

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \xrightarrow{f\otimes 1} D_{\mathrm{cris}}(W) \otimes_{K_0} B_{\mathrm{cris}} \cong W \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \tag{3.11}$$

where the isomorphisms are given by Proposition 3.2.10. Moreover, Proposition 3.2.18 implies that this map restricts to a linear map $\phi : V \longrightarrow W$. In other words, we may identify the map (3.11) as $\phi \otimes 1$. In particular, since the isomorphisms in (3.11) are Γ_K-equivariant, we recover f as the restriction of $\phi \otimes 1$ on $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\Gamma_K} \cong D_{\mathrm{cris}}(V)$. This precisely means that f is induced by ϕ via the functor D_{cris}. □

Proposition 3.2.20. *Let V be a p-adic representation of Γ_K. Let L be a finite unramified extension of K with the residue field extension l of k. Denote by Γ_L the absolute Galois group of L and by L_0 the fraction field of the ring of Witt vectors over l.*

(1) There exists a natural isomorphism of filtered isocrystals

$$D_{\mathrm{cris},K}(V) \otimes_{K_0} L_0 \cong D_{\mathrm{cris},L}(V)$$

where we set $D_{\mathrm{cris},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ and $D_{\mathrm{cris},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_L}$.

(2) V is crystalline if and only if it is crystalline as a representation of Γ_L.

Proof. We only need to prove the first statement, as the second statement immediately follows from the first statement. By definition L and L_0 are respectively unramified extensions of K and K_0 with the residue field extension l of k. Hence L and L_0 are respectively Galois over K and K_0 with $\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L_0/K_0)$. Furthermore, since the construction of B_{cris} depends only on $\mathbb{C}_K$, we have an identification

$$D_{\mathrm{cris},K}(V) = D_{\mathrm{cris},L}(V)^{\mathrm{Gal}(L/K)} = D_{\mathrm{cris},L}(V)^{\mathrm{Gal}(L_0/K_0)}.$$

Then by the Galois descent for vector spaces we obtain a natural bijective L_0-linear map

$$D_{\mathrm{cris},K}(V) \otimes_{K_0} L_0 \longrightarrow D_{\mathrm{cris},L}(V). \tag{3.12}$$

This map is evidently compatible with the natural Frobenius automorphisms on both sides induced by φ as explained in Lemma 3.2.5 and Proposition 3.2.7. Moreover, Proposition 2.4.14 and Proposition 3.2.8 together imply that the map (3.12) induces a natural L-linear isomorphism of filtered vector spaces

$$(D_{\mathrm{cris},K}(V) \otimes_{K_0} K) \otimes_K L \cong D_{\mathrm{cris},L}(V) \otimes_{L_0} L.$$

We thus deduce that the map (3.12) is an isomorphism of filtered isocrystals over L. □

Remark. Proposition 3.2.20 also holds when L is the completion of the maximal unramified extension of K. As a consequence, we have the following facts:

(1) Every unramified p-adic representation is crystalline.

(2) For a continuous character $\eta : \Gamma_K \longrightarrow \mathbb{Z}_p^{\times}$, we have $\mathbb{Q}_p(\eta) \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ if and only if there exists some $n \in \mathbb{Z}$ such that $\eta\chi^n$ is trivial on I_K.

On the other hand, Example 3.2.9 shows that Proposition 3.2.20 fails when L is a ramified extension of K. Fontaine interpreted this "failure" as a good feature of the crystalline condition, and conjectured that the crystalline condition should provide a p-adic analogue of the Néron-Ogg-Shafarevich criterion introduced in Theorem 1.1.1 of Chapter I; more precisely, Fontaine conjectured that an abelian variety A over K has good reduction if and only if the rational Tate module $V_p(A[p^\infty])$ is crystalline. Fontaine's conjecture is now known to be true by the work of Coleman-Iovita and Breuil.

We conclude this section with a discussion of a classical example which is enlightening in many ways. We assume the following technical result without proof.

Proposition 3.2.21. *The continuous map* $\log : \mathbb{Z}_p(1) \longrightarrow B_{\mathrm{dR}}^+$ *extends to a* Γ_K*-equivariant homomorphism* $\log : A_{\mathrm{inf}}[1/p]^\times \longrightarrow B_{\mathrm{dR}}^+$ *such that* $\log([p^\flat])$ *is transcendental over the fraction field of* B_{cris}.

Example 3.2.22. The *Tate curve* E_p is an elliptic curve over K with $E_p(\overline{K}) \cong \overline{K}^\times/p^{\mathbb{Z}}$ where we set $p^{\mathbb{Z}} := \{\, p^n : n \in \mathbb{Z} \,\}$. We assert that the rational Tate module $V_p(E_p[p^\infty])$ is de Rham but not crystalline. It is evident by construction that ε and $p^\flat$ form a basis of $V_p(E_p[p^\infty])$ over $\mathbb{Q}_p$. Moreover, for every $\gamma \in \Gamma_K$ we have

$$\gamma(\varepsilon) = \varepsilon^{\chi(\gamma)} \qquad \text{and} \qquad \gamma(p^\flat) = p^\flat \varepsilon^{c(\gamma)} \tag{3.13}$$

for some $c(\gamma) \in \mathbb{Z}_p$. Hence $V_p(E_p[p^\infty])$ is an extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$, and thus is de Rham by Example 2.4.5.

We aim to present a basis for $D_{\mathrm{dR}}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$. By (3.13) we find $\varepsilon \otimes t^{-1} \in D_{\mathrm{dR}}(V_p(E_p[p^\infty]))$. Let us now consider the homomorphism $\log : A_{\mathrm{inf}}[1/p]^\times \longrightarrow B_{\mathrm{dR}}^+$ as in Proposition 3.2.21 and set $u := \log([p^\flat])$. Then for $\gamma \in \Gamma_K$ we find

$$\gamma(u) = \gamma(\log([p^\flat]) = \log([\gamma(p^\flat)]) = \log([p^\flat\varepsilon^{c(\gamma)}]) = \log([p^\flat]) + c(\gamma)\log([\varepsilon]) = u + c(\gamma)t$$

by (3.13) and Lemma 2.2.20, and consequently obtain

$$\begin{aligned}\gamma(-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1) &= -\varepsilon^{\chi(\gamma)} \otimes (u + c(\gamma)t)\chi(\gamma)^{-1}t^{-1} + p^\flat\varepsilon^{c(\gamma)} \otimes 1 \\ &= -\varepsilon \otimes (ut^{-1} + c(\gamma)) + c(\gamma)\cdot(\varepsilon \otimes 1) + p^\flat \otimes 1 \\ &= -\varepsilon \otimes ut^{-1} + p^\flat \otimes 1\end{aligned}$$

by (3.13) and Theorem 2.2.21. In particular, we have $-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1 \in D_{\mathrm{dR}}(V_p(E_p[p^\infty]))$. Since the elements $\varepsilon \otimes t^{-1}$ and $-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1$ are linearly independent over B_{dR}, they form a basis for $D_{\mathrm{dR}}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$.

Let us now consider an arbitrary element $x \in D_{\mathrm{cris}}(V_p(E_p[p^\infty])) = (V_p(E_p[p^\infty]) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$. We may uniquely write $x = \varepsilon \otimes c + p^\flat \otimes d$ for some $c, d \in B_{\mathrm{cris}}$. Moreover, since we have $D_{\mathrm{cris}}(V_p(E_p[p^\infty])) \subseteq D_{\mathrm{dR}}(V_p(E_p[p^\infty]))$ there exist some $r, s \in K$ with

$$x = r\cdot(\varepsilon \otimes t^{-1}) + s\cdot(-\varepsilon \otimes ut^{-1} + p^\flat \otimes 1) = \varepsilon \otimes (r - su)t^{-1} + p^\flat \otimes s.$$

Then we find $c = (r - su)t^{-1}$, and consequently obtain $s = 0$ by Proposition 3.2.21. Therefore we deduce that every element in $D_{\mathrm{cris}}(V_p(E_p[p^\infty])) \otimes_{K_0} K$ is a K-multiple of $\varepsilon \otimes t^{-1}$. In particular, we find $\dim_{K_0} D_{\mathrm{cris}}(V_p(E_p[p^\infty])) \leq 1$, thereby concluding that $V_p(E_p[p^\infty])$ is not crystalline.

Remark. Fontaine constructed the *semistable period ring* B_{st} as the B_{cris}-subalgebra of B_{dR} generated by $\log([p^\flat])$.

CHAPTER IV

The Fargues-Fontaine curve

1. Construction

Our main objective in this section is to discuss the construction of the Fargues-Fontaine curve. The primary references are Fargues and Fontaine's survey paper [**FF12**] and Lurie's notes [**Lur**]

1.1. Untilts of a perfectoid field

Throughout this chapter, we let F be an algebraically closed perfectoid field F of characteristic p with the valuation ν_F, and write $\mathfrak{m}_F$ for the maximal ideal of $\mathcal{O}_F$. We also denote by $A_{\text{inf}} = W(\mathcal{O}_F)$ the ring of Witt vectors over $\mathcal{O}_F$, and by $W(F)$ the ring of Witt vectors over F. In addition, for every $c \in F$ we write $[c]$ for its Teichmüller lift in $W(F)$.

Definition 1.1.1. An *untilt* of F is a perfectoid field C together with a continuous isomorphism $\iota : F \simeq C^\flat$.

Example 1.1.2. The *trivial untilt* of F is the field F with the natural isomorphism $F \cong F^\flat$ given by Corollary 2.1.13 in Chapter III.

Definition 1.1.3. Let C be an untilt of F with a continuous isomorphism $\iota : F \simeq C^\flat$.

(1) We define the *sharp map* associated to C as the composition of the maps

$$F \xrightarrow[\iota]{\sim} C^\flat = \varprojlim_{x \mapsto x^p} C \longrightarrow C$$

where the last arrow is the projection to the first component.

(2) For every $c \in F$, we denote its image under the sharp map by $c^{\sharp_C}$, or often by $c^\sharp$.

(3) We define the *normalized valuation* on C to be the unique valuation ν_C with $\nu_F(c) = \nu_C(c^\sharp)$ for all $c \in F$ as given by Proposition 2.1.8 from Chapter III.

Our first goal in this subsection is to prove that every untilt of F is algebraically closed.

Lemma 1.1.4. *Let L be a complete nonarchimedean field, and let $f(x)$ be an irreducible monic polynomial over L with $f(0) \in \mathcal{O}_L$. Then $f(x)$ is a polynomial over $\mathcal{O}_L$.*

Proof. Let us choose a valuation ν_L on L. Take a finite Galois extension L' of L such that $f(x)$ factors as

$$f(x) = \prod_{i=1}^{d} (x - r_i) \qquad \text{with } r_i \in L'.$$

The valuation ν_L uniquely extends to a $\text{Gal}(L'/L)$-equivariant valuation $\nu_{L'}$ on L'. In particular, the roots r_i all have the same valuation as they belong to the same $\text{Gal}(L'/L)$-orbit. Since we have $f(0) = (-1)^d r_1 r_2 \cdots r_d \in \mathcal{O}_L$, we find that each r_i has a nonnegative valuation. Hence each coefficient of $f(x)$ has a nonnegative valuation as well. ☐

Proposition 1.1.5. *Let C be an untilt of F, and let $f(x)$ be an irreducible monic polynomial of degree d over C. For every $y \in C$, there exists an element $z \in C$ with*

$$\nu_C(y - z) \geq \nu_C(f(y))/d \qquad \text{and} \qquad \nu_C(f(z)) \geq \nu_C(p) + \nu_C(f(y)).$$

PROOF. We may replace $f(x)$ by $f(x + y)$ to assume $y = 0$. Our assertion is that there exists an element $z \in C$ with

$$\nu_C(z) \geq \nu_C(f(0))/d \qquad \text{and} \qquad \nu_C(f(z)) \geq \nu_C(p) + \nu_C(f(0)). \tag{1.1}$$

If we have $f(0) = 0$, the assertion is trivial as we can simply take $z = 0$. We henceforth assume $f(0) \neq 0$. Since F is algebraically closed, the multiplication by d is surjective on the value group of F. Hence Proposition 2.1.11 in Chapter III implies that the multiplication by d is also surjective on the value group of C. In particular, there exists an element $a \in C$ with $d\nu_C(a) = \nu_C(f(0))$. Then we can rewrite the inequalities in (1.1) as

$$\nu_C(z/a) \geq 0 \qquad \text{and} \qquad \nu_C\left(f(a \cdot (z/a))/a^d\right) \geq \nu_C(p).$$

Therefore we may replace $f(x)$ by the monic polynomial $f(a \cdot x)/a^d$ to assume $\nu_C(f(0)) = 0$. Then our assertion amounts to the existence of an element $z \in \mathcal{O}_C$ with $f(z) \in p\mathcal{O}_C$.

Lemma 1.1.4 implies that $f(x)$ is a polynomial over $\mathcal{O}_C$. In other words, we may write $f(x) = x^d + a_1x^{d-1} + \cdots + a_d$ with $a_i \in \mathcal{O}_C$. Then by Lemma 2.1.10 in Chapter III we find elements $c_i \in \mathcal{O}_F$ with $a_i - c_i^\sharp \in p\mathcal{O}_C$. Since F is algebraically closed, the polynomial $g(x) := x^d + c_1x^{d-1} + \cdots + c_d$ over $\mathcal{O}_F$ has a root α in $\mathcal{O}_F$. Now we find

$$\begin{aligned} f(\alpha^\sharp) &= (\alpha^\sharp)^d + a_1(\alpha^\sharp)^{d-1} + \cdots + a_d \\ &= (\alpha^\sharp)^d + c_1^\sharp(\alpha^\sharp)^{d-1} + \cdots + c_d^\sharp && \bmod p\mathcal{O}_C \\ &= (\alpha^d + c_1\alpha^{d-1} + \cdots + c_d)^\sharp && \bmod p\mathcal{O}_C \\ &= g(\alpha)^\sharp = 0 \end{aligned}$$

where the third identity follows from Proposition 2.1.9 in Chapter III. Hence we complete the proof by taking $z = \alpha^\sharp$. □

Proposition 1.1.6. *Every untilt of F is algebraically closed.*

PROOF. Let C be an untilt of F, and let $f(x)$ an arbitrary monic irreducible polynomial of degree d over C. We wish to show that $f(x)$ has a root in C. We may replace $f(x)$ by $p^{nd}f(x/p^n)$ for sufficiently large n to assume that $f(x)$ is a polynomial over $\mathcal{O}_C$. Let us set $y_0 := 0$ so that we have $\nu_C(f(y_0)) = \nu_C(f(0)) \geq 0$. By Proposition 1.1.5 we can inductively construct a sequence (y_n) in C with

$$\nu_C(y_{n-1} - y_n) \geq (n-1)\nu_C(p)/d \qquad \text{and} \qquad \nu_C(f(y_n)) \geq n\nu_C(p) \qquad \text{for each } n \geq 1.$$

Then the sequence (y_n) is Cauchy by construction, and thus converges to an element $y \in C$. Hence we find

$$f(y) = f\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} f(y_n) = 0,$$

thereby deducing the desired assertion. □

Remark. In order to avoid a circular reasoning, we should not deduce Proposition 1.1.6 as a special case of the tilting equivalence for perfectoid fields as stated in Chapter I, Theorem 2.2.3. In fact, the only known proof of the tilting equivalence (due to Scholze) is based on Proposition 1.1.6.

Corollary 1.1.7. *For every untilt C of F, the associated sharp map is surjective.*

We now aim to parametrize all untilts of F by certain principal ideals of A_{inf}.

Definition 1.1.8. Let C_1 and C_2 be untilts of F with continuous isomorphisms $\iota_1 : F \simeq C_1^\flat$ and $\iota_2 : F \simeq C_2^\flat$. We say that C_1 and C_2 are *equivalent* if there exists a continuous isomorphism $C_1 \simeq C_2$ such that the induced isomorphism $C_1^\flat \simeq C_2^\flat$ fits into the commutative diagram

$$\begin{array}{ccc} C_1^\flat & \xrightarrow{\sim} & C_2^\flat \\ & {\scriptstyle \iota_1}\nwarrow^{\sim} \quad {}^{\sim}\nearrow{\scriptstyle \iota_2} & \\ & F & \end{array}.$$

Example 1.1.9. Corollary 2.1.13 in Chapter III implies that the trivial untilt of F represents a unique equivalence class of characteristic p untilts of F.

Proposition 1.1.10. *Let C be a perfectoid field.*

(1) Every continuous isomorphism $F \simeq C^\flat$ induces an isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathfrak{m}_F$.

(2) Every isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathfrak{m}_F$ uniquely lifts to a continuous isomorphism $F \simeq C^\flat$.

PROOF. Let us first consider the statement (1). We regard C as an untilt of F with the given continuous isomorphism $F \simeq C^\flat$. Then Proposition 2.1.11 in Chapter III yields an element $\varpi \in F$ with $\nu_F(\varpi) = \nu_C(p) > 0$. Moreover, the continuous isomorphism $F \simeq C^\flat$ restricts to an isomorphism of valuation rings $\mathcal{O}_F \simeq \mathcal{O}_{C^\flat}$. Let us now consider the map

$$\mathcal{O}_F \xrightarrow{c \mapsto c^\sharp} \mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/p\mathcal{O}_C$$

where the second arrow is the natural projection. This map is a ring homomorphism as noted in Chapter III, Proposition 2.1.9, and is surjective by Lemma 2.1.10 in Chapter III. In addition, the kernel consists precisely of the elements $c \in \mathcal{O}_F$ with $\nu_C(c^\sharp) \geq \nu_C(p)$, or equivalently $\nu_F(c) \geq \nu_F(\varpi)$. Hence we have an induced isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ as asserted.

It remains to prove the statement (2). Since F is isomorphic to its tilt as noted in Example 1.1.9, we have an identification $\mathcal{O}_F \cong \mathcal{O}_{F^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_F$. Hence every isomorphism $\mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathfrak{m}_F$ uniquely gives rise to an isomorphism

$$\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F/\varpi\mathcal{O}_F \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C \cong \mathcal{O}_{C^\flat}$$

where the first and the third isomorphisms are given by Proposition 2.1.7 in Chapter III, and in turn lifts to a continuous isomorphism $F \simeq C^\flat$. □

Definition 1.1.11. We say that an element $\xi \in A_{\text{inf}}$ is *primitive (of degree* 1*)* if it has the form $\xi = [\varpi] - up$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. We say that a primitive element of A_{inf} is *nondegenerate* if it is not divisible by p.

Proposition 1.1.12. *Let ξ be an element in A_{inf} with the Teichmüler expansion $\xi = \sum [c_n]p^n$.*

(1) The element ξ is primitive if and only if we have $\nu_F(c_0) > 0$ and $\nu_F(c_1) = 0$.

(2) If ξ is primitive, every unit multiple of ξ in A_{inf} is primitive.

PROOF. The first statement is straightforward to verify by writing $\xi = [c_0] + p\sum [c_{n+1}^{1/p}]p^n$. The second statement then follows by the explicit multiplication rule for A_{inf}. □

Proposition 1.1.13. *Let ξ be a nondegenerate primitive element in A_{inf}. The ring $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-torsion free and p-adically complete.*

PROOF. We first verify that $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-torsion free. Consider an element $a \in A_{\text{inf}}$ such that pa is divisible by ξ. We wish to show that a is divisible by ξ. Let us write $pa = \xi b$ for some $b \in A_{\text{inf}}$. Then we have $b \in pA_{\text{inf}}$ since ξ has a nonzero image in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. Therefore we may write $b = pb'$ for some $b' \in A_{\text{inf}}$ and obtain an identity $pa = p\xi b'$, which in turn yields $a = \xi b'$ as desired.

Let us now prove that $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-adically complete. Denote by $\widehat{A_{\text{inf}}/\xi A_{\text{inf}}}$ the p-adic completion of $A_{\text{inf}}/\xi A_{\text{inf}}$. Since A_{inf} is p-adically complete, the projection $A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/\xi A_{\text{inf}}$ induces a surjective ring homomorphism

$$A_{\text{inf}} \twoheadrightarrow \widehat{A_{\text{inf}}/\xi A_{\text{inf}}} \tag{1.2}$$

by a general fact as stated in [**Sta**, Tag 0315]. It suffices to show that the kernel of this map is ξA_{inf}. Under the identification

$$\widehat{A_{\text{inf}}/\xi A_{\text{inf}}} = \varprojlim_n (A_{\text{inf}}/\xi A_{\text{inf}})/((p^n A_{\text{inf}} + \xi A_{\text{inf}})/\xi A_{\text{inf}}) \cong \varprojlim_n A_{\text{inf}}/(p^n A_{\text{inf}} + \xi A_{\text{inf}})$$

the map (1.2) coincides with the natural map

$$A_{\text{inf}} \twoheadrightarrow \varprojlim_n A_{\text{inf}}/(p^n A_{\text{inf}} + \xi A_{\text{inf}}).$$

The kernel of this map is $\bigcap_{n=1}^{\infty} (p^n A_{\text{inf}} + \xi A_{\text{inf}})$, which clearly contains ξA_{inf}. Hence we only need to show $\bigcap_{n=1}^{\infty} (p^n A_{\text{inf}} + \xi A_{\text{inf}}) \subseteq \xi A_{\text{inf}}$. Consider an arbitrary element $u \in \bigcap_{n=1}^{\infty} (p^n A_{\text{inf}} + \xi A_{\text{inf}})$. Let us choose sequences (a_n) and (b_n) in A_{inf} with $u = p^n a_n + \xi b_n$ for each $n \geq 1$. Then we have $p^n(a_n - pa_{n+1}) = \xi(b_{n+1} - b_n)$ for every $n \geq 1$. Since ξ has a nonzero image in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$, each $b_{n+1} - b_n$ must be divisible by p^n. Hence the sequence (b_n) converges to an element $b \in A_{\text{inf}}$ by the p-adic completeness of A_{inf}. As a result we find

$$u = \lim_{n\to\infty} (p^n a_n + \xi b_n) = \lim_{n\to\infty} p^n a_n + \xi \lim_{n\to\infty} b_n = \xi b,$$

thereby completing the proof. □

Definition 1.1.14. For every primitive element $\xi \in A_{\text{inf}}$, we refer to the natural projection $\theta_\xi : A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/\xi A_{\text{inf}}$ as the *untilt map* associated to ξ.

Lemma 1.1.15. *Let ξ be a nondegenerate primitive element in A_{inf}.*

(1) For every nonzero $c \in \mathcal{O}_F$, some power of p is divisible by $\theta_\xi([c])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$.
(2) For every $m \in \mathfrak{m}_F$, some power of $\theta_\xi([m])$ is divisible by p in $A_{\text{inf}}/\xi A_{\text{inf}}$.

PROOF. Let us write $\xi = [\varpi] - pu$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. For every nonzero $c \in \mathcal{O}_F$ we may write $\varpi^i = cc'$ for some $i > 0$ and $c' \in \mathcal{O}_F$, and consequently find

$$p^i = \big(\theta_\xi(u^{-1})\theta_\xi(up)\big)^i = \theta_\xi(u)^{-i}\theta_\xi([\varpi])^i = \theta_\xi(u)^{-i}\theta_\xi([c])\theta_\xi([c']).$$

Similarly, for every $m \in \mathfrak{m}_F$ we may write $m^j = \varpi \cdot b$ for some $j > 0$ and $b \in \mathcal{O}_F$, and consequently find

$$\theta_\xi([m])^j = \theta_\xi([\varpi])\theta_\xi([b]) = \theta_\xi(pu)\theta_\xi([b]) = p\theta_\xi(u)\theta_\xi([b]).$$

We thus deduce the desired assertions. □

Proposition 1.1.16. *Let ξ be a nondegenerate primitive element in A_{inf}. Take arbitrary elements $c, c' \in \mathcal{O}_F$. Then c divides c' in $\mathcal{O}_F$ if and only if $\theta_\xi([c])$ divides $\theta_\xi([c'])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$.*

PROOF. If c divides c' in $\mathcal{O}_F$, then $\theta_\xi([c])$ divides $\theta_\xi([c'])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$ by the multiplicativity of the Teichmüller lift and the map θ_ξ. Let us now assume that c does not divide c' in $\mathcal{O}_F$. We wish to show that $\theta_\xi([c])$ does not divide $\theta_\xi([c'])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$. Suppose for contradiction that there exists an element $a \in A_{\text{inf}}/\xi A_{\text{inf}}$ with $\theta_\xi([c']) = \theta_\xi([c])a$. Since we have $\nu_F(c) > \nu_F(c')$ by assumption, there exists some $m \in \mathfrak{m}_F$ with $c = mc'$. We thus find

$$\theta_\xi([c']) = \theta_\xi([c])a = \theta_\xi([c'])\theta_\xi([m])a. \tag{1.3}$$

Moreover, c' is not zero as it is not divisible by c. Hence by Lemma 1.1.15 we may write $p^n = \theta_\xi([c'])b$ for some $n > 0$ and $b \in A_{\text{inf}}/\xi A_{\text{inf}}$. Then by (1.3) we find $p^n = p^n\theta_\xi([m])a$, which in turn yields $\theta_\xi([m])a = 1$ since p is not a zero divisor in $A_{\text{inf}}/\xi A_{\text{inf}}$ by Proposition 1.1.13. However, this is impossible because the image of $\theta_\xi([m])$ under the natural map $A_{\text{inf}}/\xi A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}})$ is nilpotent by Lemma 1.1.15. □

Proposition 1.1.17. *Let ξ be a nondegenerate primitive element in A_{inf}. Every $a \in A_{\text{inf}}/\xi A_{\text{inf}}$ is a unit multiple of $\theta_\xi([c])$ for some $c \in \mathcal{O}_F$, which is uniquely determined up to unit multiple.*

PROOF. Let us first assume that a is a unit multiple of $\theta_\xi([c_1])$ and $\theta_\xi([c_2])$ for some $c_1, c_2 \in \mathcal{O}_F$. Then $\theta_\xi([c_1])$ and $\theta_\xi([c_2])$ divide each other. Hence Proposition 1.1.16 implies that c_1 and c_2 divide each other, which means that c_1 and c_2 are unit multiples of each other.

It remains to show that a is a unit multiple of $\theta_\xi([c])$ for some $c \in \mathcal{O}_F$. We may assume $a \neq 0$ as the assertion is obvious for $a = 0$. By Proposition 1.1.13 we can write $a = p^n a'$ for some $n \geq 0$ and $a' \in A_{\text{inf}}/\xi A_{\text{inf}}$ such that a' is not divisible by p. Let us write $\xi = [\varpi] - up$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. Then we have

$$a = p^n a' = \left(\theta_\xi(u^{-1})\theta_\xi(up)\right)^n a' = \theta_\xi(u)^{-1}\theta_\xi([\varpi])^n a'.$$

Hence we may replace a by a' to assume that a is not divisible by p.

We have a natural isomorphism

$$A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) = A_{\text{inf}}/([\varpi]A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_F/\varpi\mathcal{O}_F.$$

In addition, the map θ_ξ gives rise to a commutative diagram

$$\begin{array}{ccc} A_{\text{inf}} & \xrightarrow{\theta_\xi} & A_{\text{inf}}/\xi A_{\text{inf}} \\ \downarrow & & \downarrow \\ \mathcal{O}_F \cong A_{\text{inf}}/pA_{\text{inf}} & \twoheadrightarrow & A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_F/\varpi\mathcal{O}_F \end{array} \tag{1.4}$$

where the surjectivity of the bottom middle arrow follows from the surjectivity of the other arrows. Choose an element $c \in \mathcal{O}_F$ whose image under the bottom middle arrow coincides with the image of a under the second vertical arrow. Then c is not divisible by ϖ since a is not divisible by p. Therefore we may write $\varpi = cm$ for some $m \in \mathfrak{m}_F$ and obtain

$$p = \theta_\xi(u^{-1})\theta_\xi(up) = \theta_\xi(u)^{-1}\theta_\xi([\varpi]) = \theta_\xi(u)^{-1}\theta_\xi([c])\theta_\xi([m]).$$

Now the diagram (1.4) yields an element $b \in A_{\text{inf}}/\xi A_{\text{inf}}$ with

$$a = \theta_\xi([c]) + pb = \theta_\xi([c]) + b\theta_\xi(u)^{-1}\theta_\xi([c])\theta_\xi([m]) = \theta_\xi([c])\left(1 + b\theta_\xi(u)^{-1}\theta_\xi([m])\right).$$

We thus complete the proof by observing that $1 + b\theta_\xi(u)^{-1}\theta_\xi([m])$ is a unit in $A_{\text{inf}}/\xi A_{\text{inf}}$ with

$$\left(1 + b\theta_\xi(u)^{-1}\theta_\xi([m])\right)^{-1} = 1 - \left(b\theta_\xi(u)^{-1}\theta_\xi([m])\right) + \left(b\theta_\xi(u)^{-1}\theta_\xi([m])\right)^2 - \cdots$$

where the infinite sum converges by Proposition 1.1.13 and Lemma 1.1.15. □

Proposition 1.1.18. *Let ξ be a primitive element in A_{inf}, and let C_ξ denote the fraction field of $A_{\text{inf}}/\xi A_{\text{inf}}$. Then C_ξ is an untilt of F with the valuation ring $\mathcal{O}_{C_\xi} = A_{\text{inf}}/\xi A_{\text{inf}}$ and a continuous isomorphism $\iota : F \simeq C_\xi^\flat$ induced by the canonical isomorphism*

$$\mathcal{O}_F/\varpi\mathcal{O}_F \cong A_{\text{inf}}/([\varpi]A_{\text{inf}} + pA_{\text{inf}}) = A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_{C_\xi}/pC_\xi. \tag{1.5}$$

where ϖ denotes the image of ξ under the natural map $A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. Moreover, each element $c \in \mathcal{O}_F$ maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ.

Proof. Let us write $\xi = [\varpi] - up$ with $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. We also let $\mathcal{O}$ denote the ring $A_{\text{inf}}/\xi A_{\text{inf}}$. If ϖ is zero, then we have a natural isomorphism

$$\mathcal{O} = A_{\text{inf}}/\xi A_{\text{inf}} \cong A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$$

which implies that C_ξ represents the trivial untilt of F as noted in Example 1.1.9. We henceforth assume $\varpi \neq 0$.

We assert that $\mathcal{O} = A_{\text{inf}}/\xi A_{\text{inf}}$ is an integral domain. Suppose for contradiction that there exist nonzero elements $a, b \in \mathcal{O}$ with $ab = 0$. By Proposition 1.1.17 we may write $a = \theta_\xi([c])u$ for some nonzero $c \in \mathcal{O}_F$ and $u \in \mathcal{O}^\times$. In addition, by Lemma 1.1.15 we have $\theta_\xi([c])w = p^n$ for some $n > 0$ and $w \in \mathcal{O}$. Therefore we obtain an identity

$$0 = abw = \theta_\xi([c])wub = p^n ub,$$

which yields a desired contradiction by Proposition 1.1.13.

By Proposition 1.1.17 we can define a nonnegative real-valued function ν on $\mathcal{O}^\times$ which maps each $y \in \mathcal{O}^\times$ to $\nu_F(z)$ where z is an element in $\mathcal{O}_F$ such that y is a unit multiple of $\theta_\xi([z])$. Then by construction ν is a multiplicative homomorphism whose image contains the image of ν_F. In addition, for any $y_1, y_2 \in \mathcal{O}^\times$ with $\nu(y_1) \geq \nu(y_2)$ we find by Proposition 1.1.16 that y_1 is divisible y_2 in $\mathcal{O}$, and consequently obtain

$$\nu(y_1 + y_2) = \nu((y_1/y_2 + 1)y_2) = \nu(y_1/y_2 + 1) + \nu(y_2) \geq \nu(y_2) = \min(\nu(y_1), \nu(y_2)).$$

Therefore we deduce that ν is a nondiscrete valuation on $\mathcal{O}$.

We can uniquely extend ν to a valuation on C_ξ, which we also denote by ν. For every $x \in C_\xi$ we write $x = y_1/y_2$ for some $y_1, y_2 \in \mathcal{O}$ and find by Proposition 1.1.16 that $\nu(x) = \nu(y_1) - \nu(y_2)$ is nonnegative if and only if y_1 is divisible by y_2 in $\mathcal{O}$. Hence we deduce that $\mathcal{O}$ is indeed the valuation ring of C_ξ.

Since the p-th power map is surjective on $\mathcal{O}_F/\varpi\mathcal{O}_F$, it is also surjective on $\mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi}$ by the isomorphism (1.5). In addition, from the identity

$$p = \theta_\xi(u^{-1})\theta_\xi(up) = \theta_\xi(u)^{-1}\theta_\xi([\varpi])$$

we find $\nu(p) = \nu_F(\varpi) > 0$. Hence C_ξ has residue characteristic p. Furthermore, Proposition 1.1.13 implies that C_ξ is complete with respect to the valuation ν. Therefore we deduce that C_ξ is a perfectoid field.

By Proposition 1.1.10 (and its proof) the isomorphism (1.5) uniquely lifts to an isomorphism

$$\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F/\varpi\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_\xi} = \mathcal{O}_{C_\xi^\flat}$$

where the first and the third isomorphisms are given by Proposition 2.1.7 in Chapter III, and in turn lifts to a continuous isomorphism $F \simeq C_\xi^\flat$. Moreover, it is straightforward to verify that each element $c \in \mathcal{O}_F$ maps to $(\theta_\xi([c^{1/p^n}]) \in \mathcal{O}_{C_\xi^\flat}$ under the above isomorphism, and consequently maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ. Therefore we complete the proof. ∎

Proposition 1.1.19. *Let C be an untilt of F.*

(1) *There exists a surjective ring homomorphism $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ with*

$$\theta_C\left(\sum[c_n]p^n\right) = \sum c_n^\sharp p^n \qquad \text{for every } c_n \in \mathcal{O}_F.$$

(2) *Every primitive element in $\ker(\theta_C)$ generates $\ker(\theta_C)$.*

Proof. Since C is algebraically closed as noted in Proposition 1.1.6, all results from the first part of §2.2 in Chapter III remain valid with C in place of $\mathbb{C}_K$. In particular, the statement (1) is merely a restatement of Proposition 2.2.2 in Chapter III. Furthermore, Proposition 2.2.6 in Chapter III implies that $\ker(\theta_C)$ is generated by a primitive element $\xi_C := [p^\flat] - p \in A_{\text{inf}}$ where $p^\flat$ denotes an element in $\mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

Let us now consider an arbitrary primitive element $\xi \in \ker(\theta_C)$. The map θ_C induces a surjective map $\widetilde{\theta_\xi} : A_{\text{inf}}/\xi A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$. Then $\ker(\widetilde{\theta_\xi})$ is a non-maximal prime ideal as $\mathcal{O}_C$ is an integral domain but not a field. Moreover, $\ker(\widetilde{\theta_\xi})$ is a principal ideal generated by the image of ξ_C. Since A_{inf}/ξ is a valuation ring by Proposition 1.1.18, we find $\ker(\widetilde{\theta_\xi}) = 0$ and consequently deduce that ξ generates $\ker(\theta_C)$. □

Remark. In the last sentence, we used an elementary fact that every nonzero principal prime ideal of a valuation ring is maximal.

Definition 1.1.20. Given an untilt C of F, we refer to the ring homomorphism θ_C constructed in Proposition 1.1.19 as the *untilt map* of C.

Theorem 1.1.21 (Kedlaya-Liu [**KL15**], Fontaine [**Fon13**]). *There is a bijection*

$$\{\text{ equivalence classes of untilts of } F\ \} \xrightarrow{\sim} \{\text{ ideals of } A_{\text{inf}} \text{ generated by a primitive element}\}$$

which maps each untilt C of F to $\ker(\theta_C)$.

Proof. We first verify that the association is surjective. Consider an arbitrary primitive element $\xi \in A_{\text{inf}}$. By Proposition 1.1.18 it gives rise to an untilt C_ξ of F such that each $c \in \mathcal{O}_F$ maps to $\theta_\xi([c])$ under the associated sharp map. Hence Lemma 2.3.1 from Chapter II implies that the maps θ_ξ and θ_{C_ξ} coincide, thereby yielding $\xi A_{\text{inf}} = \ker(\theta_\xi) = \ker(\theta_{C_\xi})$.

It remains to show that the association is injective. Let C be an arbitrary untilt of F with a continuous isomorphism $\iota : F \simeq C^\flat$. Choose a primitive element $\omega \in \ker(\theta_C)$, which gives rise to an untilt C_ω of F with a continuous isomorphism $\iota_\omega : F \simeq C_\omega^\flat$ by Proposition 1.1.18. It suffices to show that C and C_ω are equivalent. The map θ_C induces an isomorphism $\mathcal{O}_{C_\omega} = A_{\text{inf}}/\omega A_{\text{inf}} \simeq \mathcal{O}_C$, which extends to an isomorphism $C_\omega \simeq C$. Let f denote the induced map $C_\omega^\flat \simeq C^\flat$. Then by Proposition 1.1.10 and Proposition 1.1.18 the map $f \circ \iota_\omega$ yields an isomorphism

$$\mathcal{O}_F/\varpi\mathcal{O}_F \cong A_{\text{inf}}/(pA_{\text{inf}} + \omega A_{\text{inf}}) = \mathcal{O}_{C_\omega}/p\mathcal{O}_{C_\omega} \simeq \mathcal{O}_C/p\mathcal{O}_C \tag{1.6}$$

where ϖ denotes the image of ω in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. For every $c \in \mathcal{O}_F$, this isomorphism maps the image of c in $\mathcal{O}_F/\varpi\mathcal{O}_F$ to the image of $\theta_C([c]) = c^\sharp$ in $\mathcal{O}_C/p\mathcal{O}_C$. This implies that an element $c \in \mathcal{O}_F$ is divisible by ϖ if and only if $c^\sharp$ is divisible by p, and consequently yields $\nu_F(\varpi) = \nu_C(p)$. Then the proof of Proposition 1.1.10 shows that the isomorphism (1.6) is also induced by ι. Therefore the second part of Proposition 1.1.10 yields $f \circ \iota_\omega = \iota$, which means that C and C_ω are equivalent as desired. □

Remark. The first paragraph of our proof shows that there is no conflict between Definition 1.1.14 and Definition 1.1.20.

1.2. The schematic Fargues-Fontaine curve

The main goal of this subsection is to describe the construction of the Fargues-Fontaine curve as a scheme. For the rest of this chapter, we fix a nonzero element $\varpi \in \mathfrak{m}_F$. We also denote by $Y_F = Y$ the set of equivalence classes of characteristic 0 untilts of F.

Definition 1.2.1. Let C be an untilt of F. We define the *associated absolute value* on C by

$$|x|_C := p^{-\nu_C(x)} \qquad \text{for every } x \in C,$$

and write $|C| := \{ \, |x|_C : x \in C \, \}$ for the associated absolute value group. If $C = F$ is the trivial untilt of F, we often drop the subscript to ease the notation.

Remark. Thus far we have been using valuations to describe the topology on valued fields, because valuations are convenient for topological arguments involving algebraic objects such as p-adic representations and period rings. From now on, we will use absolute values to describe the topology on perfectoid fields, because the objects of our interest will be very much analytic in nature.

Example 1.2.2. Let C be an untilt of F. Theorem 1.1.21 yields a primitive element $\xi \in A_{\text{inf}}$ which generates $\ker(\theta_C)$. If we write $\xi = [m] - up$ for some $m \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^{\times}$, we have

$$|p|_C = \left|\theta_C(u)^{-1}\theta_C([m])\right|_C = |\theta_C([m])|_C = \left|m^\sharp\right|_C = |m| \, .$$

Proposition 1.2.3. *We have an identification*

$$A_{\text{inf}}[1/p, 1/[\varpi]] = \left\{ \sum [c_n]p^n \in W(F)[1/p] : |c_n| \text{ bounded} \right\}.$$

In particular, the ring $A_{\text{inf}}[1/p, 1/[\varpi]]$ does not depend on our choice of ϖ.

Proof. Consider an arbitrary element $f = \sum [c_n]p^n \in W(F)[1/p]$. Then we have $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exists some $i > 0$ with $[\varpi^i]f = \sum [c_n\varpi^i]p^n \in A_{\text{inf}}[1/p]$, or equivalently $|c_n| \leq \left|\varpi^{-i}\right|$ for all n. □

Definition 1.2.4. Let y be an element of Y, represented by an untilt C of F.

(1) We define the *absolute value* of y by $|y| := |p|_C$.
(2) For every $f = \sum [c_n]p^n \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we define its *value at* y by

$$f(y) := \widetilde{\theta_C}(f) = \sum c_n^\sharp p^n$$

where $\widetilde{\theta_C} : A_{\text{inf}}[1/p, 1/[\varpi]] \longrightarrow C$ is the ring homomorphism which extends the untilt map $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$.

Remark. A useful heuristic idea for understanding the construction and the structure of the Fargues-Fontaine curve is that the set Y behaves in many aspects as the punctured unit disk $\mathbb{D}^* := \{ \, z \in \mathbb{C} : 0 < |z| < 1 \, \}$ in the complex plane. Here we present a couple of analogies between Y and $\mathbb{D}^*$.

(1) For each $y \in Y$ represented by an untilt C of F, its absolute value $|y| = |p|_C$ is a real number between 0 and 1. This is an analogue of the fact that every element $z \in \mathbb{D}^*$ has an absolute value between 0 and 1.
(2) Every element in $A_{\text{inf}}[1/p, 1/[\varpi]]$ is a "Laurent series in the variable p" with bounded coefficients, and gives rise to a function on Y as described in Definition 1.2.4. This is an analogue of the fact that every Laurent series $\sum a_n z^n$ over $\mathbb{C}$ with bounded coefficients defines a holomorphic function on $\mathbb{D}^*$.

Lemma 1.2.5. *Let $f = \sum [c_n]p^n$ be a nonzero element in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$, and let ρ be a real number with $0 < \rho < 1$. Then $\sup_{n\in\mathbb{Z}}(|c_n|\,\rho^n)$ exists and is attained by finitely many values of n.*

Proof. Let us take an integer n_0 with $c_{n_0} \neq 0$. Proposition 1.2.3 implies that there exists an integer $l > 0$ with $|c_n|\,\rho^n < |c_{n_0}|\,\rho^{n_0}$ for all $n > l$. In addition, there exists an integer $k < 0$ with $c_n = 0$ for all $n < k$. Therefore $\sup_{n\in\mathbb{Z}}(|c_n|\,\rho^n) = \sup_{k<n<l}(|c_n|\,\rho^n)$ exists and can only be attained by an integer n with $k < n < l$. □

Definition 1.2.6. Let ρ be a real number with $0 < \rho < 1$.

(1) We define the *Gauss ρ-norm* on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ by
$$\left|\sum [c_n]p^n\right|_\rho := \sup_{n\in\mathbb{Z}}(|c_n|\,\rho^n).$$

(2) Given an element $f = \sum [c_n]p^n \in A_{\mathrm{inf}}[1/p, 1/[\varpi]]$, we say that ρ is *generic* for f if there exists a unique $n \in \mathbb{Z}$ with $|f|_\rho = |c_n|\,\rho^n$.

Lemma 1.2.7. *Let f be an element in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$. The set*
$$S_f := \{\, \rho \in (0, 1) : \rho \textit{ is generic for } f \,\}$$
is dense in the interval $(0, 1)$.

Proof. If $\rho \in (0, 1)$ is not generic for f, then by Lemma 1.2.5 there exist some distinct integers m and n with $|f|_\rho = |c_m|\,\rho^m = |c_n|\,\rho^n$, which yields $\rho = (|c_m| / |c_n|)^{1/(n-m)}$. We thus deduce that the complement of S_f in $(0, 1)$ is countable, thereby obtaining the assertion. □

Lemma 1.2.8. *Let y be an element in Y represented by an untilt C of F. For every $f \in A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ we have $|f(y)|_C \leq |f|_{|y|}$ with equality if $|y|$ is generic for f.*

Proof. Let us write $f = \sum [c_n]p^n$ with $c_n \in F$. Then we have
$$|f(y)|_C = \left|\sum c_n^\sharp p^n\right|_C \leq \sup_{n\in\mathbb{Z}}\left(\left|c_n^\sharp\right|_C \cdot |p|_C^n\right) = \sup_{n\in\mathbb{Z}}(|c_n| \cdot |y|^n) = |f|_{|y|}\,.$$
It is evident that the inequality above becomes an equality if $|y|$ is generic for f. □

Proposition 1.2.9. *For every positive real number $\rho < 1$, the Gauss ρ-norm on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ is a multiplicative norm.*

Proof. Let f and g be arbitrary elements in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$. We wish to show
$$|f + g|_\rho \leq \max(|f|_\rho\,, |g|_\rho) \qquad \text{and} \qquad |fg|_\rho = |f|_\rho\,|g|_\rho\,.$$
Since $|F|$ is dense in the set of nonnegative real numbers, Lemma 1.2.7 implies that the set
$$S := \{\, \tau \in (0, 1) \cap |F| : \tau \text{ is generic for } f, g, f + g, \text{ and } fg \,\}$$
is dense in the interval $(0, 1)$. Hence we write $\rho = \lim_{n\to\infty} \tau_n$ for some (τ_n) in S to assume $\rho \in S$.

Take an element $c \in \mathfrak{m}_F$ with $|c| = \rho$. Then $\xi := [c]-p \in A_{\mathrm{inf}}$ is a nondegenerate primitive element, and thus gives rise to an element $y \in Y$ with $|y| = \rho$ by Proposition 1.1.13, Theorem 1.1.21, and Example 1.2.2. Then by Lemma 1.2.8 we find
$$\begin{aligned}|f + g|_\rho &= |f(y) + g(y)|_C \leq \max(|f(y)|_C\,, |g(y)|_C) = \max(|f|_\rho\,, |g|_\rho),\\ |fg|_\rho &= |f(y)g(y)|_C = |f(y)|_C\,|g(y)|_C = |f|_\rho\,|g|_\rho\,.\end{aligned}$$
Therefore we complete the proof. □

Definition 1.2.10. Let $[a, b]$ be a closed subinterval of $(0, 1)$. We write

$$Y_{[a,b]} := \{ \, y \in Y : a \leq |y| \leq b \, \},$$

and define the *ring of holomorphic functions on* $Y_{[a,b]}$, denoted by $B_{[a,b]}$, to be the completion of $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ with respect to the Gauss a-norm and the Gauss b-norm.

Lemma 1.2.11. *Let* $[a, b]$ *be a closed subinterval of* $(0, 1)$*, and let* f *be an element in* $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$*. We have* $|f|_\rho \leq \sup(|f|_a, |f|_b)$ *for all* $\rho \in [a, b]$*.*

Proof. Let us write $f = \sum [c_n] p^n$ for some $c_n \in F$. Then we have

$$\begin{aligned} |c_n| \rho^n &\leq |c_n| b^n \leq |f|_b && \text{for all } n \geq 0, \\ |c_n| \rho^n &\leq |c_n| a^n \leq |f|_a && \text{for all } n < 0. \end{aligned}$$

Hence we deduce the desired assertion. □

Remark. Since $|F|$ is dense in $(0, \infty)$, we find $\sup\limits_{|y|=\rho} (|f(y)|_C) = |f|_\rho$ for all $\rho \in |F| \cap (0, 1)$ by Lemma 1.2.7 and Lemma 1.2.8. Hence we may regard Lemma 1.2.11 as an analogue of the maximum modulus principle for holomorphic functions on $\mathbb{D}^*$.

Proposition 1.2.12. *Let* $[a, b]$ *be a closed subinterval of* $(0, 1)$*. The ring* $B_{[a,b]}$ *is the completion of* $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ *with respect to all Gauss* ρ*-norms with* $\rho \in [a, b]$*.*

Proof. Lemma 1.2.11 implies that a sequence (f_n) in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ is Cauchy with respect to the Gauss a-norm and the Gauss b-norm if and only if it is Cauchy with respect to the Gauss ρ-norm for all $\rho \in [a, b]$. □

Corollary 1.2.13. *For any* $a, b, a', b' \in \mathbb{R}$ *with* $[a, b] \subseteq [a', b'] \subseteq (0, 1)$*, we have* $B_{[a',b']} \subseteq B_{[a,b]}$*.*

Definition 1.2.14. We define the *ring of holomorphic functions on* Y by

$$B_F := \varprojlim B_{[a,b]}$$

where the transition maps are the natural inclusions given by Corollary 1.2.13. We often write B instead of B_F to ease the notation.

Remark. It is not hard to see that a formal sum $\sum [c_n] p^n$ with $c_n \in F$ converges in B if and only if it satisfies

$$\limsup_{n>0} |c_n|^{1/n} \leq 1 \qquad \text{and} \qquad \lim_{n \to \infty} |c_{-n}|^{1/n} = 0.$$

This is an analogue of the fact that a Laurent series $\sum a_n z^n$ over $\mathbb{C}$ converges on $\mathbb{D}^*$ if and only if it satisfies

$$\limsup_{n>0} |a_n|^{1/n} \leq 1 \qquad \text{and} \qquad \lim_{n \to \infty} |a_{-n}|^{1/n} = 0.$$

However, an arbitrary element in B may not admit a unique "Laurent series expansion" in p, whereas every holomorphic function on $\mathbb{D}^*$ admits a unique Laurent series expansion.

Lemma 1.2.15. *Let* $\eta : R_1 \longrightarrow R_2$ *be a continuous homomorphism of normed rings.*

(1) The map η *uniquely extends to a continuous ring homomorphism* $\widehat{\eta} : \widehat{R_1} \longrightarrow \widehat{R_2}$ *where* $\widehat{R_1}$ *and* $\widehat{R_2}$ *respectively denote the completions of* R_1 *and* R_2*.*
(2) The homomorphism $\widehat{\eta}$ *is a homeomorphism if* η *is a homeomorphism.*

Proof. This is an immediate consequence of an elementary fact from analysis. □

Proposition 1.2.16. *Let C be a characteristic 0 untilt of F. The untilt map θ_C uniquely extends to a surjective continuous open ring homomorphism $\widehat{\theta_C} : B \twoheadrightarrow C$.*

PROOF. The map θ_C uniquely extends to a surjective ring homomorphism

$$\widetilde{\theta_C} : A_{\text{inf}}[1/p, 1/[\varpi]] \twoheadrightarrow \mathcal{O}_C[1/p] = C.$$

Let us set $\rho := |p|_C$. Then $\widetilde{\theta_C}$ uniquely extends to a surjective continuous ring homomorphism $\widehat{\widetilde{\theta_C}} : B_{[\rho,\rho]} \twoheadrightarrow C$ by Lemma 1.2.8 and Lemma 1.2.15. Moreover, $\widehat{\widetilde{\theta_C}}$ is open by the open mapping theorem. Take $\widehat{\theta_C}$ to be the restriction of $\widehat{\widetilde{\theta_C}}$ on B. By construction $\widehat{\theta_C}$ is a surjective continuous open map which extends $\widetilde{\theta_C}$. Since the uniqueness is evident by the continuity, we deduce the desired assertion. □

Definition 1.2.17. Let y be an element in Y, represented by an untilt C of F.

(1) We refer to the map $\widehat{\theta_C}$ given by Proposition 1.2.16 as the *evaluation map* at y.
(2) For every $f \in B$, we define its *value at* y by $f(y) := \widehat{\theta_C}(f)$.

Proposition 1.2.18. *The Frobenius automorphism of F uniquely lifts to a continuous automorphism φ on B.*

PROOF. Let $\widetilde{\varphi_F}$ denote the Frobenius automorphism of $W(F)$. By construction we have

$$\widetilde{\varphi_F}\left(\sum [c_n]p^n\right) = \sum [c_n^p]p^n \qquad \text{for all } c_n \in F. \tag{1.7}$$

Then Proposition 1.2.3 implies that $\widetilde{\varphi_F}$ restricts to an automorphism on $A_{\text{inf}}[1/p, 1/[\varpi]]$. Moreover, by (1.7) we find

$$|\widetilde{\varphi_F}(f)|_{\rho^p} = |f|_\rho^p \qquad \text{for all } f \in A_{\text{inf}}[1/p, 1/[\varpi]] \text{ and } \rho \in (0,1). \tag{1.8}$$

Consider an arbitrary closed interval $[a,b] \subseteq (0,1)$, and choose a real number $r \in [a,b]$. By Lemma 1.2.15 and (1.8) the map $\widetilde{\varphi_F}$ on $A_{\text{inf}}[1/p, 1/[\varpi]]$ uniquely extends to a continuous ring isomorphism $\varphi_{[r,r]} : B_{[r,r]} \simeq B_{[r^p,r^p]}$. In addition, the identity (1.8) implies that a sequence (f_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ is Cauchy with respect to the Gauss a-norm and the Gauss b-norm if and only if the sequence $(\widetilde{\varphi_F}(f_n))$ is Cauchy with respect to the Gauss a^p-norm and the Gauss b^p-norm. Since $\widetilde{\varphi_F}$ is bijective, we deduce that $\varphi_{[r,r]}$ restricts to a continuous ring isomorphism $\varphi_{[a,b]} : B_{[a,b]} \simeq B_{[a^p,b^p]}$ with an inverse given by the restriction of $\varphi_{[r,r]}^{-1}$ on $B_{[a^p,b^p]}$. It is evident by construction that $\varphi_{[a,b]}$ is an extension of $\widetilde{\varphi_F}$.

By our discussion in the preceding paragraph, the map $\widetilde{\varphi_F}$ on $A_{\text{inf}}[1/p, 1/[\varpi]]$ extends to a continuous isomorphism

$$\varphi : B = \varprojlim B_{[a,b]} \simeq \varprojlim B_{[a^p,b^p]} = B.$$

Moreover, the uniqueness of φ is evident by the continuity. Therefore we obtain the desired assertion. □

Definition 1.2.19. We refer to the map φ constructed in Proposition 1.2.18 as the *Frobenius automorphism* of B, and define the *schematic Fargues-Fontaine curve* as the scheme

$$X_F := \text{Proj}\left(\bigoplus_{n \geq 0} B^{\varphi = p^n}\right).$$

We often simply write X instead of X_F to ease the notation.

1.3. The adic Fargues-Fontaine curve

In this subsection, we describe another incarnation of the Fargues-Fontaine curve using the language of adic spaces developed by Huber in [**Hub93**] and [**Hub94**]. Our goal for this subsection is twofold: introducing a new perspective for the construction of the Fargues-Fontaine curve, and providing an exposition on some related theories. Our discussion will be cursory, as we won't use any results from this section in the subsequent sections.

Definition 1.3.1. Let R be a topological ring.

(1) We say that a subset S of R is *bounded* if for every open neighborhood U of 0 there exists an open neighborhood V of 0 with $VS \subseteq U$.
(2) We say that an element $f \in R$ is *power-bounded* if the set $\{ f^n : n \geq 0 \}$ is bounded, and denote by R° the subring of power-bounded elements in R.
(3) We say that R is a *Huber ring* if there exists an open subring R_0, called a *ring of definition*, on which the induced topology is generated by a finitely generated ideal.
(4) If R is a Huber ring, we say that R is *uniform* if R° is a ring of definition.

Example 1.3.2. We present some important examples of uniform Huber rings.

(1) Every ring R with the discrete topology is a uniform Huber ring with $R^\circ = R$, as its topology is generated by the zero ideal.
(2) Every nonarchimedean field L is a uniform Huber ring with $L^\circ = \mathcal{O}_L$, as the topology on $\mathcal{O}_L$ is generated by the ideal $m\mathcal{O}_L$ for any m in the maximal ideal.
(3) The ring A_{inf} is a uniform Huber ring with $A_{\text{inf}}^\circ = A_{\text{inf}}$ and the topology generated by the ideal $pA_{\text{inf}} + [\varpi]A_{\text{inf}}$.

Definition 1.3.3. A *Huber pair* is a pair (R, R^+) which consists of a Huber ring R and its open and integrally closed subring $R^+ \subseteq R^\circ$.

Proposition 1.3.4. *For every Huber ring R, the subring R° is open and integrally closed.*

Definition 1.3.5. Let R be a topological ring.

(1) A map $v : R \longrightarrow T \cup \{ 0 \}$ for some totally ordered abelian group T is called a *continuous multiplicative valuation* if it satisfies the following properties:
   (i) $v(0) = 0$ and $v(1) = 1$.
   (ii) For all $r, s \in R$ we have $v(rs) = v(r)v(s)$ and $v(r + s) \leq \max(v(r), v(s))$.
   (iii) For every $\tau \in T$ the set $\{ r \in R : v(r) < \tau \}$ is open in R.
(2) We say that two continuous multiplicative valuations v and w on R are *equivalent* if there exists an isomorphism of totally ordered monoids $\delta : v(R) \cup \{ 0 \} \simeq w(R) \cup \{ 0 \}$ with $\delta(v(r)) = w(r)$ for all $r \in R$.
(3) We define the *valuation spectrum* of R, denoted by $\text{Spv}(R)$, to be the set of equivalence classes of continuous multiplicative valuations on R.
(4) Given $r \in R$ and $x \in \text{Spv}(R)$, we define the *value of r at x* by $|r(x)| := v(r)$ where v is any representative of x.

Remark. Our terminology in (1) slightly modifies Huber's original terminology *continuous valuation* in order to avoid any potential confusion after extensively using the term valuation in the additive notation.

Proposition 1.3.6. *Let v and w be continuous multiplicative valuations on a topological ring R. Then v and w are equivalent if and only if for all $r, s \in R$ the inequality $v(r) \leq v(s)$ amounts to the inequality $w(r) \leq w(s)$.*

Definition 1.3.7. For a Huber pair (R, R^+), we define its *adic spectrum* by

$$\mathrm{Spa}(R, R^+) := \left\{ x \in \mathrm{Spv}(R) : |f(x)| \leq 1 \text{ for all } f \in R^+ \right\}$$

endowed with the topology generated by subsets of the form

$$\mathcal{U}(f/g) := \left\{ x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0 \right\} \qquad \text{for some } f, g \in R.$$

Example 1.3.8. We are particularly interested in the set

$$\mathcal{Y} := \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \backslash \{ x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) : |p[\varpi](x)| = 0 \},$$

which we call the *perfectoid punctured unit disk*. Let us describe two types of points on $\mathcal{Y}$.

Let y be an element in Y, represented by an untilt C of F. Consider a nonnegative real valued function v_y on A_{inf} defined by $v_y(f) := |f(y)|_C = |\theta_C(f)|_C$ for every $f \in A_{\mathrm{inf}}$. It is evident by construction that v_y is a continuous multiplicative valuation on A_{inf} with $v_y(f) \leq 1$ for all $f \in A_{\mathrm{inf}}$. In addition, we have $v_y(p) = |p|_C \neq 0$ and $v_y([\varpi]) = |\varpi| \neq 0$. Hence v_y gives rise to a point in $\mathcal{Y}$, which we denote by $\widetilde{y}$.

Let ρ be a real number with $0 < \rho < 1$. By Proposition 1.2.9 the Gauss ρ-norm on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ restricts to a continuous multiplicative valuation on A_{inf} with $|f|_\rho \leq 1$ for all $f \in A_{\mathrm{inf}}$. In addition, we have $|p|_\rho = \rho \neq 0$ and $|[\varpi]|_\rho = |\varpi| \neq 0$. Hence the Gauss ρ-norm on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ gives rise to a point in $\mathcal{Y}$, which we denote by γ_ρ.

Remark. Interested readers may find some informative illustrations of $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ and $\mathcal{Y}$ in Scholze's Berkeley lectures [**SW20**, §12].

Definition 1.3.9. Let (R, R^+) be a Huber pair. A *rational subset* of $\mathrm{Spa}(R, R^+)$ is a subset of the form

$$\mathcal{U}(T/g) := \left\{ x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0 \text{ for all } f \in T \right\}$$

for some $g \in R$ and some nonempty finite set $T \subseteq R$ such that TR is open in R.

Example 1.3.10. We say that a subset of $\mathcal{Y}$ is *distinguished* if it has the form

$$\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]} := \left\{ x \in \mathcal{Y} : \left|[\varpi^i](x)\right| \leq |p(x)| \leq \left|[\varpi^j](x)\right| \right\}$$

for some $i, j \in \mathbb{Z}[1/p]$ with $0 < j \leq i$. Every distinguished subset of $\mathcal{Y}$ is a rational subset of $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$; indeed, we have an identification

$$\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]} = \left\{ x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) : \left|[\varpi^{i+j}](x)\right|, \left|p^2(x)\right| \leq \left|[\varpi^j]p(x)\right| \neq 0 \right\} = \mathcal{U}(T_{[i,j]}/[\varpi^j]p)$$

where $T_{i,j} := \left\{ [\varpi^{i+j}], p^2 \right\}$ generates an open ideal in A_{inf}. In particular, every distinguished subset of $\mathcal{Y}$ is open in $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$.

Let us describe some points on each $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ in line with our discussion in Example 1.3.8. For an element $y \in Y$, we have $\widetilde{y} \in \mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ if and only if y is an element of $Y_{[|\varpi|^i, |\varpi|^j]}$. For a real number ρ with $0 < \rho < 1$, we have $\gamma_\rho \in \mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ if and only if ρ belongs to the interval $[|\varpi|^i, |\varpi|^j]$.

Remark. We can extend our discussion above by defining the *absolute value* for an arbitrary point $x \in \mathcal{Y}$. We say that a valuation is of rank 1 if it takes values in the set of positive real numbers. It is a fact that x admits a unique maximal generization $x^{\max}$ of rank 1. We define the absolute value of x by

$$|x| := |\varpi|^{\frac{\log(|p(x^{\max})|)}{\log(|[\varpi](x^{\max})|)}}.$$

Let us now consider $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ of $\mathcal{Y}$ for some $i, j \in \mathbb{Z}[1/p]$. Since $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ is open in $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ as noted above, the point x lies in $\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}$ if and only if $x^{\max}$ does, which amounts to having $|x| \in [|\varpi|^i, |\varpi|^j]$.

Proposition 1.3.11. *Let (R, R^+) be a Huber pair, and write $\mathcal{S} := \mathrm{Spa}(R, R^+)$. Consider a rational subset $\mathcal{U} := \mathcal{U}(T/g)$ for some $g \in R$ and some nonempty finite set $T \subseteq R$ such that TR is open in R.*

(1) There exists a map of Huber pairs $(R, R^+) \longrightarrow (\mathcal{O}_{\mathcal{S}}(\mathcal{U}), \mathcal{O}^+_{\mathcal{S}}(\mathcal{U}))$ for some complete Huber ring $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$ with the following properties:
   (i) The induced map $\mathrm{Spa}(\mathcal{O}_{\mathcal{S}}(\mathcal{U}), \mathcal{O}^+_{\mathcal{S}}(\mathcal{U})) \longrightarrow \mathcal{S}$ yields a homeomorphism onto $\mathcal{U}$.
   (ii) It is universal for maps of Huber pairs $(R, R^+) \longrightarrow (Q, Q^+)$ such that the induced map $\mathrm{Spa}(Q, Q^+) \longrightarrow \mathcal{S}$ factors over $\mathcal{U}$.

(2) If R is uniform such that the topology on R° is given by a finitely generated ideal I, then $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$ is given by the completion of $R[1/g]$ with respect to the ideal generated by I and the set $T' := \{ f/g : f \in T \}$.

Definition 1.3.12. Let (R, R^+) be a Huber pair, and write $\mathcal{S} := \mathrm{Spa}(R, R^+)$. We define the presheaves $\mathcal{O}_{\mathcal{S}}$ and $\mathcal{O}^+_{\mathcal{S}}$ on $\mathcal{S}$ by

$$\mathcal{O}_{\mathcal{S}}(\mathcal{W}) := \varprojlim_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \mathcal{U} \text{ rational}}} \mathcal{O}_{\mathcal{S}}(\mathcal{U}) \quad \text{and} \quad \mathcal{O}^+_{\mathcal{S}}(\mathcal{W}) := \varprojlim_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \mathcal{U} \text{ rational}}} \mathcal{O}^+_{\mathcal{S}}(\mathcal{U}) \quad \text{for all open } \mathcal{W} \subseteq \mathcal{S}$$

where $\mathcal{O}_{\mathcal{S}}(\mathcal{U})$ and $\mathcal{O}^+_{\mathcal{S}}(\mathcal{U})$ for each rational subset $\mathcal{U}$ of $\mathcal{S}$ are given by Proposition 1.3.11. We refer to $\mathcal{O}_{\mathcal{S}}$ as the *structure presheaf* of $\mathcal{S}$.

Remark. The ring $\mathcal{O}^+_{\mathcal{S}}(\mathcal{W})$ is in general not open in $\mathcal{O}_{\mathcal{S}}(\mathcal{W})$.

Example 1.3.13. Let us write $\mathcal{S} := \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$. We assert that $\mathcal{Y}$ is an open subset of $\mathcal{S}$ with $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}) \cong B$. The set $\mathcal{Y}$ is covered by the distinguished subsets; indeed, as both $[\varpi]$ and p are topologically nilpotent in A_{inf}, for every $x \in \mathcal{Y}$ there exist some positive real numbers $i, j \in \mathbb{Z}[1/p]$ with $\left|[\varpi^i](x)\right| \leq |p(x)|$ and $\left|p^{1/j}(x)\right| \leq |[\varpi](x)|$, or equivalently $\left|[\varpi^i](x)\right| \leq |p(x)| \leq \left|[\varpi^j](x)\right|$. Since distinguished subsets of $\mathcal{Y}$ are (open) rational subsets of $\mathcal{S}$ as noted in Example 1.3.10, we deduce that $\mathcal{Y}$ is an open subset of $\mathcal{S}$ with

$$\mathcal{O}_{\mathcal{S}}(\mathcal{Y}) = \varprojlim \mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}) \tag{1.9}$$

where the limit is taken over all distinguished subsets of $\mathcal{Y}$.

Consider arbitrary numbers $i, j \in \mathbb{Z}[1/p]$ with $0 < j \leq i$. In light of (1.9) it suffices to establish an identification

$$\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]}) \cong B_{[|\varpi|^i, |\varpi|^j]}. \tag{1.10}$$

Proposition 1.3.11 and Example 1.3.2 together imply that $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[|\varpi|^i, |\varpi|^j]})$ is the completion of $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ with respect to the ideal I generated by the set $T := \left\{ p, [\varpi], [\varpi^i]/p, p/[\varpi^j] \right\}$. Moreover, the ideal I is generated by $[\varpi^i]/p$ and $p/[\varpi^j]$ as we have $p = (p/[\varpi^j]) \cdot [\varpi^j]$ and $[\varpi] = ([\varpi^i]/p)^r \cdot p^r \cdot (1/[\varpi])^s$ for some positive integers r and s. It is then straightforward to verify that the I-adic topology on $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ coincides with the topology induced by the Gauss $|\varpi|^i$-norm and the Gauss $|\varpi|^j$-norm. Therefore we obtain the identification (1.10) as desired.

Definition 1.3.14. We say that a Huber pair (R, R^+) is *sheafy* if the structure presheaf on $\mathrm{Spa}(R, R^+)$ is a sheaf.

Proposition 1.3.15. *Let (R, R^+) be a Huber pair, and write $\mathcal{S} := \mathrm{Spa}(R, R^+)$.*

(1) For every open $\mathcal{W} \subseteq \mathcal{S}$ we have

$$\mathcal{O}^+_{\mathcal{S}}(\mathcal{W}) = \{ f \in \mathcal{O}_{\mathcal{S}}(\mathcal{W}) : |f(x)| \leq 1 \text{ for all } x \in \mathcal{W} \}.$$

(2) The presheaf $\mathcal{O}^+_{\mathcal{S}}$ is a sheaf if (R, R^+) is sheafy.

Definition 1.3.16. Let R be a Huber ring.

(1) We say that R is *Tate* if it contains a topologically nilpotent unit.
(2) We say that R is *strongly noetherian* if for every $n \geq 0$ the Tate algebra
$$R\langle u_1, \cdots, u_n\rangle := \Big\{ \sum a_{i_1,\cdots,i_n} u_1^{i_1} \cdots u_n^{i_n} \in R[[u_1, \cdots, u_n]] : \lim a_{i_1,\cdots,i_n} = 0 \Big\}$$
is noetherian.

Theorem 1.3.17 (Huber [**Hub94**])**.** *A Huber pair (R, R^+) is sheafy if R is Tate and strongly noetherian.*

Theorem 1.3.18 (Kedlaya [**Ked16**])**.** *For every closed interval $[a, b] \subseteq (0, 1)$ the topological ring $B_{[a,b]}$ is a Tate and strongly noetherian Huber ring.*

Definition 1.3.19. An *adic space* is a topological space $\mathcal{S}$ together with a sheaf $\mathcal{O}_{\mathcal{S}}$ of topological rings and a continuous multiplicative valuation v_x on $\mathcal{O}_{\mathcal{S},x}$ for each $x \in \mathcal{S}$ such that $\mathcal{S}$ is locally of the form $\mathrm{Spa}(R, R^+)$ for some sheafy Huber pair (R, R^+).

Example 1.3.20. By Example 1.3.13, Theorem 1.3.17 and Theorem 1.3.18 we deduce that distinguished subsets of $\mathcal{Y}$ are noetherian adic spaces, and in turn find that $\mathcal{Y}$ is a locally noetherian adic space. In addition, for every closed interval $[a, b] \subseteq (0, 1)$ we see that
$$\mathcal{Y}_{[a,b]} := \bigcup_{[|\varpi|^i,|\varpi|^j]\subseteq[a,b]} \mathcal{Y}_{[|\varpi|^i,|\varpi|^j]}$$
is a locally noetherian adic space with $\mathcal{O}_{\mathcal{Y}_{[a,b]}}(\mathcal{Y}_{[a,b]}) = B_{[a,b]}$.

Proposition 1.3.21. *Every morphism of Huber pairs $g : (R, R^+) \longrightarrow (Q, Q^+)$ induces a map of presheaves $\mathcal{O}_{\mathcal{S}} \longrightarrow g_*\mathcal{O}_{\mathcal{T}}$ where we write $\mathcal{S} := \mathrm{Spa}(R, R^+)$ and $\mathcal{T} := \mathrm{Spa}(Q, Q^+)$.*

Example 1.3.22. Let ϕ denote the automorphism of $\mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ induced by the Frobenius automorphism of A_{inf}. It is evident by construction that $\mathcal{Y}$ is stable under ϕ. In addition, by Example 1.3.13 and Proposition 1.3.21 we get an induced automorphism on $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \cong B$ which is easily seen to coincide with φ.

Let us choose $c \in (1/p, p) \cap \mathbb{Q}$. For every $n \in \mathbb{Z}$, we set
$$\mathcal{V}_n := \mathcal{Y}_{[|\varpi|^{1/p^n},|\varpi|^{c/p^n}]} \qquad \text{and} \qquad \mathcal{W}_n := \mathcal{Y}_{[|\varpi|^{c/p^n},|\varpi|^{c/p^{n+1}}]}.$$
Arguing as in Example 1.3.13, we find that $\mathcal{Y}$ is covered by such sets. In addition, we have $\phi(\mathcal{V}_n) = \mathcal{V}_{n-1}$ and $\phi(\mathcal{W}_n) = \mathcal{W}_{n-1}$ for all $n \in \mathbb{Z}$. Therefore the action of ϕ on $\mathcal{Y}$ is properly discontinuous, and consequently yields the quotient space
$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}}.$$
Moreover, $\mathcal{X}$ is covered by (the isomorphic images of) $\mathcal{V}_0$ and $\mathcal{W}_0$, which are noetherian adic spaces as noted in Example 1.3.20. Hence $\mathcal{X}$ is a noetherian adic space with $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) \cong B^{\varphi=1}$.

Definition 1.3.23. We refer to the noetherian adic space $\mathcal{X}$ constructed in Example 1.3.22 as the *adic Fargues-Fontaine curve.*

Theorem 1.3.24 (Kedlaya-Liu [**KL15**])**.** *There exists a natural morphism of locally ringed spaces $h : \mathcal{X} \longrightarrow X$ such that the pullback along h induces an equivalence*
$$h^* : \mathrm{Bun}_X \xrightarrow{\sim} \mathrm{Bun}_{\mathcal{X}}$$
where Bun_X and $\mathrm{Bun}_{\mathcal{X}}$ respectively denote the categories of vector bundles on X and $\mathcal{X}$.

Remark. Theorem 1.3.24 is often referred to as "GAGA for the Fargues-Fontaine curve". By Theorem 1.3.24, studying the schematic Fargues-Fontaine curve is essentially equivalent to studying the adic Fargues-Fontaine curve.

2. Geometric structure

In this section we establish some fundamental geometric properties of the Fargues-Fontaine curve. Our discussion will show that the Fargues-Fontaine curve is geometrically very akin to proper curves over $\mathbb{Q}_p$. In addition, our discussion will provide a number of new perspectives towards several constructions from Chapter III. The primary references for this section are Fargues and Fontaine's survey paper [**FF12**] and Lurie's notes [**Lur**]

2.1. Legendre-Newton polygons

We begin by introducing a crucial tool for studying the structure of the ring B.

Definition 2.1.1. Let $\log_p$ denote the real logarithm base p.

(1) Given an element $f \in B$, we define the *Legendre-Newton polygon* of f as the function $\mathcal{L}_f : (0, \infty) \longrightarrow \mathbb{R} \cup \{ \infty \}$ given by

$$\mathcal{L}_f(s) := -\log_p\left(|f|_{p^{-s}}\right) \qquad \text{for all } s \in (0, \infty).$$

(2) Given a closed interval $[a, b] \subseteq (0, 1)$ and an element $f \in B_{[a,b]}$, we define the *Legendre-Newton* $[a, b]$*-polygon* of f as the function $\mathcal{L}_{f,[a,b]} : [-\log_p(b), -\log_p(a)] \longrightarrow \mathbb{R} \cup \{ \infty \}$ given by

$$\mathcal{L}_{f,[a,b]}(s) := -\log_p\left(|f|_{p^{-s}}\right) \qquad \text{for all } s \in [-\log_p(b), -\log_p(a)].$$

Remark. With notations as in Example 1.3.8, we may write $\mathcal{L}_f(s) = -\log_p\left(\left|f(\gamma_{p^{-s}})\right|\right)$ for all $f \in B$ and $s \in (0, \infty)$.

Lemma 2.1.2. *Given any elements $f, g \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we have*

$$\mathcal{L}_{fg}(s) = \mathcal{L}_f(s) + \mathcal{L}_g(s) \quad \text{and} \quad \mathcal{L}_{f+g}(s) \geq \min(\mathcal{L}_f(s), \mathcal{L}_g(s)) \quad \text{for all } s \in (0, \infty).$$

Proof. This is an immediate consequence of Proposition 1.2.9. □

Our main goal in this subsection is to prove that Legendre-Newton polygons are indeed polygons with decreasing integer slopes.

Definition 2.1.3. Let g be a piecewise linear function defined on an interval $I \subseteq \mathbb{R}$.

(1) We say that g is *concave* if the slopes are decreasing, and *convex* if the slopes are increasing.
(2) We write $\partial_- g$ and $\partial_+ g$ respectively for the left and right derivatives of g.

Example 2.1.4. Let $f = \sum [c_n]p^n$ be a nonzero element in $A_{\text{inf}}[1/p, 1/[\varpi]]$. Its *Newton polygon* is defined as the lower convex hull the points $(n, \nu_F(c_n)) \in \mathbb{R}^2$, which we may regard as a convex piecewise linear function on $(0, \infty)$.

Lemma 2.1.5. *Given a nonzero element $f = \sum [c_n]p^n \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we have*

$$\mathcal{L}_f(s) = \inf_{n \in \mathbb{Z}} (\nu_F(c_n) + ns) \qquad \textit{for every } s \in (0, \infty).$$

Proof. This is obvious by definition. □

Remark. By Lemma 2.1.5 it is not hard to verify that $\mathcal{L}_f$ coincides with the Legendre transform of the Newton polygon of f.

Example 2.1.6. Let ξ be a primitive element in A_{inf} with the Teichmüller expansion $\xi = \sum [c_n]p^n$. By Proposition 1.1.12 we have

$$\mathcal{L}_\xi(s) = \min(\nu_F(c_0), \nu_F(c_1) + s) = \min(\nu_F(c_0), s) \qquad \text{for all } s \in (0, \infty).$$

Proposition 2.1.7. *Let $f = \sum[c_n]$ be a nonzero element in $A_{\inf}[1/p, 1/[\varpi]]$.*

(1) $\mathcal{L}_f$ is a concave piecewise linear function with integer slopes.

(2) For each $s \in (0, \infty)$, the one-sided derivatives $\partial_-\mathcal{L}_f(s)$ and $\partial_+\mathcal{L}_f(s)$ are respectively given by the minimum and maximum elements of the set

$$T_s := \{\, n \in \mathbb{Z} : \mathcal{L}_f(s) = \nu_F(c_n) + ns \,\}.$$

Proof. Fix a real number $s > 0$. Lemma 2.1.5 and Lemma 1.2.5 together imply that T_s is finite. Let l and r respectively denote the minimum and maximum elements of T_s. By construction we have

$$\nu_F(c_l) + ls = \nu_F(c_r) + rs \leq \nu_F(c_n) + ns \qquad \text{for all } n \in \mathbb{Z} \tag{2.1}$$

where equality holds if and only if n belongs to T_s. It suffices to show that for all sufficiently small $\epsilon > 0$ we have

$$\mathcal{L}_f(s+\epsilon) = \mathcal{L}_f(s) + l\epsilon \qquad \text{and} \qquad \mathcal{L}_f(s-\epsilon) = \mathcal{L}_f(s) - r\epsilon. \tag{2.2}$$

Let us consider the first identity in (2.2). Take $k < 0$ with $c_n = 0$ for all $n \leq k$, and set

$$\delta_1 := \inf_{n<l} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)}{l-n} \right) = \inf_{k<n<l} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)}{l-n} \right).$$

Then we have $\delta_1 > 0$ as the inequality in (2.1) is strict for all $n < l$. Let ϵ be a real number with $0 < \epsilon < \delta_1$. For every $n < l$ we find $\epsilon(l-n) < \delta_1(l-n) \leq (\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)$ and consequently obtain

$$\nu_F(c_l) + l(s+\epsilon) < \nu_F(c_n) + n(s+\epsilon).$$

In addition, for every $n > l$ we have

$$\nu_F(c_l) + l(s+\epsilon) \leq \nu_F(c_n) + ns + l\epsilon < \nu_F(c_n) + n(s+\epsilon)$$

where the first inequality follows from (2.1). Therefore we obtain

$$\mathcal{L}_f(s+\epsilon) = \inf_{n\in\mathbb{Z}} (\nu_F(c_n) + n(s+\epsilon)) = \nu_F(c_l) + l(s+\epsilon) = \mathcal{L}_f(s) + l\epsilon.$$

We now consider the second identity in (2.2). Proposition 1.2.3 implies that there exists $\lambda \in \mathbb{R}$ with $\nu_F(c_n) > \lambda$ for all $n \in \mathbb{Z}$. Let us set

$$u := \frac{\nu_F(c_r) - \lambda}{s/2} + r \quad \text{and} \quad \delta_2 := \inf_{r<n<u} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_r) + rs)}{n-r} \right).$$

Then we have $\delta_2 > 0$ as the inequality in (2.1) is strict for all $n > r$. Let ϵ be a real number with $0 < \epsilon < \min(s/2, \delta_2)$. For every $n > u$ we find

$$\nu_F(c_r) - \nu_F(c_n) < \nu_F(c_r) - \lambda = (u-r)s/2 < (n-r)(s-\epsilon)$$

and consequently obtain

$$\nu_F(c_r) + r(s-\epsilon) < \nu_F(c_n) + n(s-\epsilon).$$

In addition, we get the same inequality for every $n < r$ by arguing as in the preceding paragraph. Therefore we deduce

$$\mathcal{L}_f(s-\epsilon) = \inf_{n\in\mathbb{Z}} (\nu_F(c_n) + n(s-\epsilon)) = \nu_F(c_r) + r(s-\epsilon) = \mathcal{L}_f(s) - r\epsilon,$$

thereby completing the proof. □

Remark. In light of the remark after Lemma 2.1.5, we can alternatively deduce Proposition 2.1.7 from a general fact that the Legendre transform of a convex piecewise linear function with integer breakpoints is a concave piecewise linear function with integer slopes.

Lemma 2.1.8. *Let (f_n) be a Cauchy sequence in $A_{\text{inf}}[1/p, 1/[\varpi]]$ with respect to the Gauss p^{-s}-norm for some $s > 0$. Assume that (f_n) does not converge to 0. Then the sequences $(\mathcal{L}_{f_n}(s))$, $(\partial_-\mathcal{L}_{f_n}(s))$, and $(\partial_+\mathcal{L}_{f_n}(s))$ are all eventually constant.*

Proof. The sequence $\left(|f_n|_{p^{-s}}\right)$ converges in $\mathbb{R}$. Let us set

$$a := \lim_{n\to\infty} \mathcal{L}_{f_n}(s) = -\lim_{n\to\infty} \log_p\left(|f_n|_{p^{-s}}\right),$$

and take an integer $u > 0$ with

$$\mathcal{L}_{f_n - f_u}(s) = -\log_p\left(|f_n - f_u|_{p^{-s}}\right) > 2a \qquad \text{and} \qquad \mathcal{L}_{f_n}(s) < 2a \qquad \text{for all } n \geq u.$$

For every $n \geq u$, since both $\mathcal{L}_{f_u}$ and $\mathcal{L}_{f_n - f_u}$ are continuous, we may find some $\delta_n > 0$ with

$$\mathcal{L}_{f_n - f_u}(s + \epsilon) > 2a > \mathcal{L}_{f_u}(s + \epsilon) \qquad \text{for all } \epsilon \in (-\delta_n, \delta_n),$$

and consequently obtain $\mathcal{L}_{f_u}(s + \epsilon) = \mathcal{L}_{f_n}(s + \epsilon)$ for all $\epsilon \in (-\delta_n, \delta_n)$ by Lemma 2.1.2. This implies that for every $n \geq u$ we have

$$\mathcal{L}_{f_n}(s) = \mathcal{L}_{f_u}(s), \qquad \partial_-\mathcal{L}_{f_n}(s) = \partial_-\mathcal{L}_{f_u}(s), \qquad \partial_+\mathcal{L}_{f_n}(s) = \partial_+\mathcal{L}_{f_u}(s).$$

Hence we deduce the desired assertion. □

Proposition 2.1.9. *Let $[a, b]$ be a closed subinterval of $(0, 1)$, and let (f_n) be a Cauchy sequence in $A_{\text{inf}}[1/p, 1/[\varpi]]$ with respect to the Gauss a-norm and the Gauss b-norm. Assume that (f_n) does not converge to 0 with respect to either the Gauss a-norm or the Gauss b-norm. Then the sequence of functions $(\mathcal{L}_{f_n})$ is eventually constant on $[-\log_p(b), -\log_p(a)]$.*

Proof. Let us write $l := -\log_p(b)$ and $r := -\log_p(a)$. Without loss of generality we may assume that each f_n is not zero. In addition, by symmetry we may assume that f_n does not converge to 0 with respect to the Gauss b-norm. Then Lemma 2.1.8 yields $\alpha, \beta \in \mathbb{R}$ and $u \in \mathbb{Z}$ such that we have $\mathcal{L}_{f_n}(l) = \alpha$ and $\partial_+\mathcal{L}_{f_n}(l) = \beta$ for all $n > u$. Since each $\mathcal{L}_{f_n}$ is concave and piecewise linear by Proposition 2.1.7, we set $\omega := \max(\alpha, \alpha + \beta(r - l))$ and find

$$\mathcal{L}_{f_n}(s) \leq \alpha + \beta(s - l) \leq \omega \qquad \text{for all } n > u \text{ and } s \in [l, r]. \tag{2.3}$$

Moreover, Lemma 1.2.11 (or Proposition 1.2.12) implies that the sequence (f_n) converges with respect to all Gauss ρ-norms with $\rho \in [a, b]$, thereby yielding an integer $u' > u$ with $|f_n - f_{u'}|_\rho < p^{-\omega}$ for all $n > u'$ and $\rho \in [a, b]$, or equivalently

$$\mathcal{L}_{f_n - f_{u'}}(s) > \omega \qquad \text{for all } n > u' \text{ and } s \in [l, r].$$

Hence by Lemma 2.1.2 and (2.3) we find

$$\mathcal{L}_{f_n}(s) = \mathcal{L}_{f_{u'}}(s) \qquad \text{for all } n > u' \text{ and } s \in [l, r].$$

thereby deducing the desired assertion. □

Proposition 2.1.10. *Let $[a, b]$ be a closed subinterval of $(0, 1)$. For every nonzero $f \in B_{[a,b]}$, the function $\mathcal{L}_{f,[a,b]}$ is concave and piecewise linear with integer slopes.*

Proof. Take a sequence (f_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ which converges to f with respect to the Gauss a-norm and the Gauss b-norm. By Proposition 1.2.12 we have

$$\mathcal{L}_{f,[a,b]}(s) = \lim_{n\to\infty} \mathcal{L}_{f_n}(s) \qquad \text{for all } s \in [-\log_p(b), -\log_p(a)].$$

Since f is not zero, the assertion follows by Proposition 2.1.7 an Proposition 2.1.9. □

Remark. For a holomorphic function g on the annulus $\mathbb{D}^*_{[a,b]} := \{ z \in \mathbb{C} : a \leq |z| \leq b \}$, the Hadamard three-circle theorem asserts that the function $\mathcal{M}_g : [\ln(a), \ln(b)] \longrightarrow \mathbb{R}$ defined by $\mathcal{M}_g(r) := \ln\left(\sup_{|z|=e^r} (|g(z)|) \right)$ for all $r \in [\ln(a), \ln(b)]$ is convex. In light of the remark after Lemma 1.2.11 we may consider Proposition 2.1.10 as an analogue of the Hadamard three-circle theorem.

Corollary 2.1.11. *For every nonzero $f \in B$, the Legendre-Newton polygon $\mathcal{L}_f$ is a concave piecewise linear function with integer slopes.*

Remark. Corollary 2.1.11 suggests that we can define the Newton polygon of f as the Legendre transform of $\mathcal{L}_f$.

Example 2.1.12. Let f be an invertible element in B. By Lemma 2.1.2 we find

$$\mathcal{L}_f(s) = \mathcal{L}_1(s) - \mathcal{L}_{f^{-1}}(s) = -\mathcal{L}_{f^{-1}}(s) \qquad \text{for all } s \in (0, \infty).$$

Since both $\mathcal{L}_f$ and $\mathcal{L}_{f^{-1}}$ are concave piecewise linear functions as noted in Corollary 2.1.11, we deduce that $\mathcal{L}_f$ is linear.

Remark. In fact, it is not hard to prove that a nonzero element $f \in B$ is invertible if and only if $\mathcal{L}_f$ is linear.

Let us present some important applications of the Legendre-Newton polygons.

Definition 2.1.13. For every $n \in \mathbb{Z}$, we refer to the ring $B^{\varphi=p^n}$ as the *Frobenius eigenspace* of B with eigenvalue p^n.

Lemma 2.1.14. *Given an element $f \in B$, we have*

$$|\varphi(f)|_{\rho^p} = |f|_\rho^p \qquad \textit{and} \qquad |pf|_\rho = \rho\, |f|_\rho \qquad \textit{for all } \rho \in (0, 1).$$

Proof. If f is an element in $A_{\text{inf}}[1/p, 1/[\varpi]]$, the assertion is evident by construction. The assertion for the general case then follows by continuity. □

Proposition 2.1.15. *The Frobenius eigenspace $B^{\varphi=p^n}$ is trivial for every $n < 0$.*

Proof. Suppose for contradiction that $B^{\varphi=p^n}$ contains a nonzero element f. By Lemma 2.1.14 we have

$$p\mathcal{L}_f(s) = \mathcal{L}_{\varphi(f)}(ps) = \mathcal{L}_{p^n f}(ps) = nps + \mathcal{L}_f(ps) \qquad \text{for all } s > 0.$$

Since $\mathcal{L}_f$ is a concave piecewise linear function by Corollary 2.1.11, we find

$$p\partial_+\mathcal{L}_f(s) = np + p\partial_+\mathcal{L}_f(ps) \leq np + p\partial_+\mathcal{L}_f(s) \qquad \text{for all } s > 0, \tag{2.4}$$

thereby obtaining a contradiction as desired. □

Remark. A similar argument shows that $\mathcal{L}_f$ is linear for every nonzero $f \in B^{\varphi=1}$. In Proposition 3.1.6 we will build on this fact to prove that $B^{\varphi=1}$ is naturally isomorphic to $\mathbb{Q}_p$.

Proposition 2.1.16. *Let $[a, b]$ be a closed subinterval of $(0, 1)$, and let f be a nonzero element in $B_{[a,b]}$. Then we have $|f|_\rho \neq 0$ for every $\rho \in [a, b]$.*

Proof. Proposition 2.1.10 implies that $\mathcal{L}_{f,[a,b]}(-\log_p(\rho)) = -\log_p\left(|f|_\rho\right)$ is finite for every $\rho \in [a, b]$, thereby yielding the desired assertion. □

Corollary 2.1.17. *For every closed interval $[a, b] \subseteq (0, 1)$ the ring $B_{[a,b]}$ is an integral domain.*

Proof. This is an immediate consequence of Proposition 1.2.9 and Proposition 2.1.16. □

2.2. Divisors and zeros of functions

In this subsection we define the notion of divisors on Y for elements in B.

Definition 2.2.1. A *divisor* on Y is a formal sum $\sum_{y\in Y} n_y \cdot y$ with $n_y \in \mathbb{Z}$ such that for every closed interval $[a,b] \subseteq (0,1)$ the set $Z_{[a,b]} := \left\{ y \in Y_{[a,b]} : n_y \neq 0 \right\}$ is finite.

Remark. Definition 2.2.1 is comparable with the definition of Weil divisors on locally noetherian integral schemes as given in [**Sta**, Tag 0BE2].

Lemma 2.2.2. *Let f and g be elements in B. Assume that f is divisible by g in $B_{[a,b]}$ for every closed interval $[a,b] \subseteq (0,1)$. Then f is divisible by g in B.*

Proof. For every $n \geq 2$ we may write $f = gh_n$ for some $h_n \in B_{[1/n,1-1/n]}$. Then by Corollary 1.2.13 and Corollary 2.1.17 we find that h_n takes a constant value for all $n \geq 2$. Hence we get an element $h \in B$ with $h = h_n$ for all $n \geq 2$, thereby obtaining the desired assertion. □

Proposition 2.2.3. *Let y be an element in Y, represented by an untilt C of F. Every $f \in B$ with $f(y) = 0$ is divisible by every primitive element $\xi \in \ker(\theta_C)$.*

Proof. Consider an arbitrary closed interval $[a,b] \subseteq (0,1)$ with $y \in Y_{[a,b]}$. By Lemma 2.2.2 it suffices to prove that f is divisible by ξ in $B_{[a,b]}$. Take a sequence (f_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ which converges to f with respect to the Gauss a-norm and the Gauss b-norm. By Corollary 1.1.7 we may write $f_n(y) = c_n^\sharp$ for some $c_n \in F$. Then we have

$$\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \left|c_n^\sharp\right|_C = \lim_{n\to\infty} |f_n(y)|_C = |f(y)|_C = 0,$$

and consequently find that the sequence $([c_n])$ converges to 0 with respect to the Gauss a-norm and the Gauss b-norm. Hence we may replace (f_n) by $(f_n - [c_n])$ to assume $f_n(y) = 0$ for all $n > 0$.

Let $\widetilde{\theta_C} : A_{\text{inf}}[1/p, 1/[\varpi]] \longrightarrow C$ be the ring homomorphism which extends the untilt map θ_C. Proposition 1.1.19 implies that ξ generates $\ker(\widetilde{\theta_C})$. We may thus write $f_n = \xi g_n$ for some $g_n \in A_{\text{inf}}[1/p, 1/[\varpi]]$. Then for every $\rho \in [a,b]$ we use Proposition 1.2.9 to find

$$\lim_{n\to\infty} |g_{n+1} - g_n|_\rho = \frac{1}{|\xi|_\rho} \cdot \lim_{n\to\infty} |\xi(g_{n+1} - g_n)|_\rho = \frac{1}{|\xi|_\rho} \cdot \lim_{n\to\infty} |f_{n+1} - f_n|_\rho = 0,$$

which means that the sequence (g_n) is Cauchy with respect to the Gauss ρ-norm. Therefore the sequence (g_n) defines an element $g \in B_{[a,b]}$ with $f = \xi g$. □

Remark. By Corollary 1.1.7 we may write $p = (p^\flat)^\sharp$ for some $p^\flat \in \mathfrak{m}_F$, which is uniquely determined up to unit multiple. Then we obtain a primitive element $[p^\flat] - p \in \ker(\theta_C)$, and consequently find an expression $f = ([p^\flat] - p)g$ for some $g \in B$ by Proposition 2.2.3. This is an analogue of the fact that a holomorphic function f on $\mathbb{D}^*$ with a zero at $z_0 \in \mathbb{D}^*$ can be written in the form $f = (z - z_0)g$ for some holomorphic function g on $\mathbb{D}^*$.

Corollary 2.2.4. *Let C be a characteristic 0 untilt of F. Every primitive element $\xi \in \ker(\theta_C)$ generates $\ker(\widehat{\theta_C})$.*

Remark. Let $[a,b]$ be a closed subinterval of $(0,1)$ with $|p|_C \in [a,b]$. By the proof of Proposition 1.2.16 the untilt map θ_C extends to a surjective continuous ring homomorphism $\widehat{\widehat{\theta_C}} : B_{[a,b]} \twoheadrightarrow C$. Then we can similarly show that every primitive element $\xi \in \ker(\theta_C)$ generates $\ker(\widehat{\widehat{\theta_C}})$.

Proposition 2.2.5. *Let C be a characteristic 0 untilt of F, and let $\theta_C[1/p] : A_{\text{inf}}[1/p] \longrightarrow C$ be the ring homomorphism which extends the untilt map θ_C. Then we have*

$$A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j = \ker(\theta_C[1/p])^j \qquad \text{for all } j \geq 1.$$

Proof. The assertion for $j = 1$ follows by observing that $\widehat{\theta_C}$ restricts to $\theta_C[1/p]$. Let us now proceed by induction on j. We only need to show $A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j \subseteq \ker(\theta_C[1/p])^j$, since the reverse containment is obvious by the fact that $\widehat{\theta_C}$ restricts to $\theta_C[1/p]$. Let a be an arbitrary element in $A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j$, and choose a primitive element $\xi \in \ker(\theta_C)$. Then ξ generates both $\ker(\widehat{\theta_C})$ and $\ker(\theta_C[1/p])$ by Corollary 2.2.4 and Proposition 1.1.19. Hence we may write $a = \xi^j b$ for some $b \in B$. In addition, since we have

$$A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^j \subseteq A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})^{j-1} = \ker(\theta_C[1/p])^{j-1}$$

by the induction hypothesis, there exists some $c \in A_{\text{inf}}[1/p]$ with $a = \xi^{j-1}c$. We then find

$$0 = a - a = \xi^j b - \xi^{j-1}c = \xi^{j-1}(\xi b - c),$$

and consequently obtain $c = \xi b$ by Corollary 2.1.17. This implies $c \in A_{\text{inf}}[1/p] \cap \ker(\widehat{\theta_C})$, and in turn yields $c \in \ker(\theta_C[1/p])$ by the assertion for $j = 1$ that we have already established. Therefore we deduce $a = \xi^{j-1}c \in \ker(\theta_C[1/p])^j$ as desired. □

Definition 2.2.6. Let y be an element in Y, represented by an untilt C of F. We define the *de Rham local ring at y* by

$$B_{\text{dR}}^+(y) := \varprojlim_j A_{\text{inf}}[1/p]/\ker(\theta_C[1/p])^j$$

where $\theta_C[1/p] : A_{\text{inf}}[1/p] \longrightarrow C$ is the ring homomorphism which extends the untilt map θ_C.

Proposition 2.2.7. *Let y be an element in Y, represented by an untilt C of F.*

(1) *The ring $B_{\text{dR}}^+(y)$ is a complete discrete valuation ring with C as the residue field.*
(2) *Every primitive element in $\ker(\theta_C)$ is a uniformizer of $B_{\text{dR}}^+(y)$.*
(3) *There exists a natural isomorphism*

$$B_{\text{dR}}^+(y) \cong \varprojlim_j B/\ker(\widehat{\theta_C})^j.$$

Proof. Since C is algebraically closed as noted in Proposition 1.1.6, all results from the first part of §2.2 in Chapter III remain valid with C in place of $\mathbb{C}_K$. Hence the statements (1) and (2) follow from Proposition 2.2.12 in Chapter III and Proposition 1.1.19.

It remains to verify the statement (3). Let $\theta_C[1/p] : A_{\text{inf}}[1/p] \twoheadrightarrow C$ be the surjective ring homomorphism which extends the untilt map θ_C, and choose a primitive element $\xi \in \ker(\theta_C)$. Then ξ generates both $\ker(\widehat{\theta_C})$ and $\ker(\theta_C[1/p])$ by Corollary 2.2.4 and Proposition 1.1.19. Hence we get a natural map

$$B_{\text{dR}}^+(y) = \varprojlim_j A_{\text{inf}}[1/p]/\xi^j A_{\text{inf}}[1/p] \longrightarrow \varprojlim_j B/\xi^j B = \varprojlim_j B/\ker(\widehat{\theta_C})^j \tag{2.5}$$

which is easily seen to be injective by Proposition 2.2.5. Moreover, since we have

$$A_{\text{inf}}[1/p]/\xi A_{\text{inf}}[1/p] \cong C \cong B/\xi B,$$

the map (2.5) is surjective by a general fact as stated in [**Sta**, Tag 0315]. We thus deduce that the natural map (2.5) is an isomorphism, thereby completing the proof. □

Definition 2.2.8. Let f be a nonzero element in B. We define its *order of vanishing* at $y \in Y$ to be its valuation in $B_{\mathrm{dR}}^{+}(y)$, denoted by $\mathrm{ord}_y(f)$.

Remark. The element y gives rise to a point $\widetilde{y} \in \mathcal{Y}$ as described in Example 1.3.8. With Proposition 2.2.7 and our discussion in §1.3 we can show that $B_{\mathrm{dR}}^{+}(y)$ is the completed local ring at $\widetilde{y}$. In this sense, Definition 2.2.8 agrees with the usual definition for order of vanishing.

Example 2.2.9. Let ξ be a nondegenerate primitive element in A_{inf}. Theorem 1.1.21 implies that ξ vanishes at a unique element $y_\xi \in Y$. Then we have

$$\mathrm{ord}_y(\xi) = \begin{cases} 1 & \text{for } y = y_\xi, \\ 0 & \text{for } y \neq y_\xi. \end{cases}$$

Lemma 2.2.10. *Let f and g be nonzero elements in B. Then we have*

$$\mathrm{ord}_y(fg) = \mathrm{ord}_y(f) + \mathrm{ord}_y(g) \qquad \textit{for all } y \in Y.$$

Proof. This is evident by definition. □

Proposition 2.2.11. *Let f be a nonzero element in B. For every closed interval $[a, b] \subseteq (0, 1)$, the set $Z_{[a,b]} := \left\{ y \in Y_{[a,b]} : \mathrm{ord}_y(f) \neq 0 \right\}$ is finite.*

Proof. Let us write $l := -\log_p(b)$ and $r := -\log_p(a)$. We also set $n := \partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r)$, which is a nonnegative integer by Corollary 2.1.11. Since we have $\mathrm{ord}_y(f) \geq 0$ for all $y \in Y$, it suffices to show

$$\sum_{y \in Z_{[a,b]}} \mathrm{ord}_y(f) \leq n. \tag{2.6}$$

Suppose for contradiction that this inequality fails. By Proposition 2.2.3, Example 2.2.9 and Lemma 2.2.10 we may write

$$f = \xi_1 \xi_2 \cdots \xi_{n+1} g \tag{2.7}$$

for some $g \in B$ and primitive elements $\xi_1, \cdots, \xi_{n+1} \in A_{\mathrm{inf}}$ such that each ξ_i vanishes at a unique element $y_i \in Y_{[a,b]}$. Then Example 1.2.2 and Example 2.1.6 together imply that for each $i = 1, \cdots, n+1$ we have

$$\mathcal{L}_{\xi_i}(s) = \begin{cases} s & \text{for } s \leq -\log_p(|y_i|), \\ -\log_p(|y_i|) & \text{for } s > -\log_p(|y_i|). \end{cases}$$

Hence we obtain

$$\partial_- \mathcal{L}_{\xi_i}(l) - \partial_+ \mathcal{L}_{\xi_i}(r) = 1 - 0 = 1 \qquad \text{for each } i = 1, \cdots, n+1.$$

In addition, by Corollary 2.1.11 we have $\partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r) \geq 0$. Therefore we use Lemma 2.1.2 and (2.7) to find

$$\begin{aligned} n &= \partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r) \\ &= \sum_{i=1}^{n+1} \left(\partial_- \mathcal{L}_{\xi_i}(l) - \partial_+ \mathcal{L}_{\xi_i}(r) \right) + \left(\partial_- \mathcal{L}_g(l) - \partial_+ \mathcal{L}_g(r) \right) \\ &\geq n + 1, \end{aligned}$$

thereby obtaining a contradiction as desired. □

Remark. It turns out that the inequality (2.6) is indeed an equality.

Definition 2.2.12. For every $f \in B$, we define its *associated divisor* on Y by

$$\mathrm{Div}(f) := \sum_{y \in Y} \mathrm{ord}_y(f) \cdot y.$$

2.3. The logarithm and untilts

In this subsection, we define and study the logarithms of elements in the multiplicative group $1 + \mathfrak{m}_F$. For the rest of this section we write $\mathfrak{m}_F^* := \mathfrak{m}_F \setminus \{ 0 \}$.

Proposition 2.3.1. *There exists a group homomorphism* $\log : 1 + \mathfrak{m}_F \longrightarrow B^{\varphi=p}$ *with*

$$\log(\varepsilon) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \qquad \text{for every } \varepsilon \in 1 + \mathfrak{m}_F. \tag{2.8}$$

Proof. Given arbitrary $\varepsilon \in 1 + \mathfrak{m}_F$ and $\rho \in (0,1)$, we write $[\varepsilon] - 1 = \sum [c_n] p^n$ with $c_n \in \mathcal{O}_F$ to find

$$|[\varepsilon] - 1|_\rho \leq \max(|c_0|, \rho) = \max(|\varepsilon - 1|, \rho) < 1.$$

Hence we obtain a map $\log : 1 + \mathfrak{m}_F \longrightarrow B$ satisfying (2.8). It then follows that log is a group homomorphism by the identity of formal power series $\log(xy) = \log(x) + \log(y)$. Furthermore, as φ is continuous by construction, for every $\varepsilon \in 1 + \mathfrak{m}_F$ we find

$$\varphi(\log(\varepsilon)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)^n}{n} = \log(\varepsilon^p) = p \log(\varepsilon),$$

thereby completing the proof. □

Remark. We will see in Proposition 3.1.8 that log is a $\mathbb{Q}_p$-linear isomorphism.

Definition 2.3.2. We refer to the map $\log : 1 + \mathfrak{m}_F \longrightarrow B^{\varphi=p}$ constructed in Proposition 2.3.1 as the *logarithm* on $1 + \mathfrak{m}_F$.

Proposition 2.3.3. *Let C be a characteristic 0 untilt of F, and let $\mathfrak{m}_C$ denote the maximal ideal of $\mathcal{O}_C$. There exists a commutative diagram*

$$\begin{array}{ccc} 1 + \mathfrak{m}_F & \xrightarrow{\ \log\ } & B^{\varphi=p} \\ {\scriptstyle \varepsilon \mapsto \varepsilon^\sharp} \downarrow & & \downarrow {\scriptstyle \widehat{\theta_C}} \\ 1 + \mathfrak{m}_C & \xrightarrow[\log_{\mu_{p^\infty}}]{} & C \end{array} \tag{2.9}$$

where all maps are group homomorphisms.

Proof. Let c be an arbitrary element in $\mathcal{O}_F$. By Proposition 2.1.9 in Chapter III, there exists some $a \in \mathcal{O}_C$ with $c^\sharp - 1 = (c-1)^\sharp + pa$. If c belongs to $1 + \mathfrak{m}_F$, then we have

$$\left| c^\sharp - 1 \right|_C \leq \max\left(\left| (c-1)^\sharp \right|_C, |pa|_C \right) = \max(|c-1|, |pa|_C) < 1$$

and in turn obtain $c^\sharp \in 1 + \mathfrak{m}_C$. Conversely, if $c^\sharp$ belongs to $1 + \mathfrak{m}_C$, then we have

$$|c - 1| = \left| (c-1)^\sharp \right|_C \leq \max\left(\left| c^\sharp - 1 \right|_C, pa \right) < 1$$

and consequently obtain $c \in 1 + \mathfrak{m}_F$. Therefore in light of Corollary 1.1.7 we deduce that $1 + \mathfrak{m}_F$ maps onto $1 + \mathfrak{m}_C$ under the sharp map.

Since the map $\widehat{\theta_C}$ is continuous by construction, for every $\varepsilon \in 1 + \mathfrak{m}_F$ we have

$$\widehat{\theta_C}(\log(\varepsilon)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\widehat{\theta_C}([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varepsilon^\sharp - 1)^n}{n} = \log_{\mu_{p^\infty}}(\varepsilon^\sharp)$$

where the last identity follows by Example 3.3.5 in Chapter II. Moreover, as C is algebraically closed by Proposition 1.1.6, the map $\log_{\mu_{p^\infty}}$ is a surjective homomorphism by Proposition 3.3.6 in Chapter II. Therefore we obtain the commutative diagram (2.9) as desired. □

Proposition 2.3.4. *For every $\varepsilon \in 1 + \mathfrak{m}_F^*$, the element*

$$\xi_\varepsilon := \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}] \in A_{\text{inf}}$$

is a nondegenerate primitive element which divides $[\varepsilon] - 1$ but not $[\varepsilon^{1/p}] - 1$.

Proof. Let us write $k := \mathcal{O}_F/\mathfrak{m}_F$ for the residue field of F, and $W(k)$ for the ring of Witt vectors over k. In addition, for every $c \in \mathcal{O}_F$ we denote by $\overline{c}$ its image under the natural map $\mathcal{O}_F \twoheadrightarrow k$, and by $[\overline{c}]$ the Teichmüller lift of $\overline{c}$ in $W(k)$. Lemma 2.3.1 from Chapter II yields a homomorphism $\pi : A_{\text{inf}} \longrightarrow W(k)$ with

$$\pi\left(\sum [c_n]p^n\right) = \sum [\overline{c_n}]p^n \qquad \text{for all } c_n \in \mathcal{O}_F.$$

We then find $\pi(\xi_\varepsilon) = p$ by observing $\overline{\varepsilon^{1/p}} = \overline{\varepsilon}^{1/p} = 1$, and consequently obtain a Teichmüller expansion

$$\xi_\varepsilon = [m_0] + [m_1 + 1]p + \sum_{n \geq 2} [m_n]p^n \qquad \text{with } m_n \in \mathfrak{m}_F.$$

Since we have $|m_0| < 1$ and $|m_1 + 1| = 1$, we deduce by Proposition 1.1.12 that ξ_ε is a primitive element in A_{inf}. Moreover, ξ_ε is nondegenerate as we have

$$m_0 = 1 + \varepsilon^{1/p} + \cdots + \varepsilon^{(p-1)/p} = \frac{\varepsilon - 1}{\varepsilon^{1/p} - 1} \neq 0.$$

It is also evident that ξ_ε divides $[\varepsilon] - 1$. On the other hand, ξ_ε does not divide $[\varepsilon^{1/p}] - 1$, since otherwise $\xi_\varepsilon = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}]$ should divide p, yielding a contradiction by Proposition 1.1.13. □

Proposition 2.3.5. *For every $\varepsilon \in 1 + \mathfrak{m}_F^*$, there exists some $y_\varepsilon \in Y$ with $\operatorname{ord}_{y_\varepsilon}(\log(\varepsilon)) = 1$.*

Proof. Proposition 2.3.4 allows us to write $[\varepsilon] - 1 = \xi_\varepsilon([\varepsilon^{1/p}] - 1)$ for some nondegenerate primitive element $\xi_\varepsilon \in A_{\text{inf}}$ which does not divide $[\varepsilon^{1/p}] - 1$. Then by Example 2.2.9 and Lemma 2.2.10 we find an element $y_\varepsilon \in Y$ with $\operatorname{ord}_{y_\varepsilon}([\varepsilon] - 1) = 1$. This means that the image of $[\varepsilon] - 1$ in $B_{\text{dR}}^+(y_\varepsilon)$ is a uniformizer. The assertion then follows from the fact that $\log(\varepsilon)$ is divisible by $[\varepsilon] - 1$ but not by $([\varepsilon] - 1)^2$. □

Proposition 2.3.6. *There exists a bijection $Y \xrightarrow{\sim} (1 + \mathfrak{m}_F^*)/\mathbb{Z}_p^\times$ which maps the equivalence class of an untilt C of F to the $\mathbb{Z}_p^\times$-orbit of elements $\varepsilon_C \in 1 + \mathfrak{m}_F^*$ with $\varepsilon_C^\sharp = 1$ and $\left(\varepsilon_C^{1/p}\right)^\sharp \neq 1$.*

Proof. Let y be an arbitrary element in Y, represented by an untilt C of F. Choosing an element $\varepsilon_C \in 1 + \mathfrak{m}_F^*$ with $\varepsilon_C^\sharp = 1$ and $\left(\varepsilon_C^{1/p}\right)^\sharp \neq 1$ amounts to choosing a system of primitive p-power roots of unity in $C^\flat \simeq F$. Such a system exists uniquely up to $\mathbb{Z}_p^\times$-multiple by Proposition 1.1.6.

Let us now consider an arbitrary element $\varepsilon \in 1 + \mathfrak{m}_F^*$. Proposition 2.3.4 yields a nondegenerate primitive element $\xi_\varepsilon \in A_{\text{inf}}$ which divides $[\varepsilon] - 1$ but not $[\varepsilon^{1/p}] - 1$. Then by Theorem 1.1.21 we get an untilt C_ε of F with $\varepsilon^\sharp = 1$ and $\left(\varepsilon^{1/p}\right)^\sharp \neq 1$. Moreover, for every untilt C of F with $\varepsilon^\sharp = 1$ and $\left(\varepsilon^{1/p}\right)^\sharp \neq 1$, we have

$$0 = \frac{\varepsilon^\sharp - 1}{\left(\varepsilon^{1/p}\right)^\sharp - 1} = \frac{\theta_C([\varepsilon] - 1)}{\theta_C([\varepsilon^{1/p}] - 1)} = \theta_C(\xi_\varepsilon)$$

and consequently find by Proposition 1.1.19 and Theorem 1.1.21 that C and C_ε are equivalent. Therefore we deduce that ε is the image of a unique element in Y. □

Definition 2.3.7. Let φ_F denote the Frobenius automorphism of F.

(1) Given an untilt C of F with a continuous isomorphism $\iota : C^\flat \simeq F$, we define its *Frobenius twist* $\phi(C)$ as the perfectoid field C with the isomorphism $\varphi_F^n \circ \iota$.
(2) We define the *Frobenius action* on Y as the map $\phi : Y \to Y$ induced by Frobenius twists.

Lemma 2.3.8. *For every characteristic* 0 *untilt* C *of* F *we have* $\widehat{\theta_{\phi(C)}} = \widehat{\theta_C} \circ \varphi$

PROOF. The identity is evident on $A_{\text{inf}}[1/p, 1/[\varpi]]$ by construction. The assertion then follows by continuity. □

Remark. In Example 1.3.22 we described the Frobenius action ϕ on $\mathcal{Y}$. By Lemma 2.3.8 it is straightforward to check that the map $Y \longrightarrow \mathcal{Y}$ given by Example 1.3.8 is compatible with the Frobenius actions on Y and $\mathcal{Y}$.

Proposition 2.3.9. *Let* f *be a nonzero element in* $B^{\varphi=p^n}$ *for some* $n \geq 0$*. Then we have* $\text{ord}_y(f) = \text{ord}_{\phi(y)}(f)$ *for all* $y \in Y$*.*

PROOF. Let C be an untilt of F which represents y. By corollary 2.2.4 there exists a primitive element ξ which generates $\ker(\widehat{\theta_C})$. It is then straightforward to check by Proposition 1.1.12 that $\varphi(\xi)$ is a primitive element in A_{inf}. Moreover, we have $\varphi(\xi) \in \ker(\widehat{\theta_{\phi(C)}})$ by Lemma 2.3.8. Let us write $i := \text{ord}_y(f)$ and $j := \text{ord}_{\phi(y)}(f)$. By Proposition 2.2.7 we may write

$$f = \xi^i g = \varphi(\xi)^j h \qquad \text{with } g, h \in B.$$

Then we have $f = p^{-n}\varphi(f) = \varphi(\xi)^i \cdot p^{-n} g$ and consequently find $i \leq j$. Similarly, we have $f = \varphi^{-1}(\varphi(f)) = p^n \varphi^{-1}(f) = \xi^j \cdot p^n h$ and consequently find $i \geq j$. Therefore we deduce $i = j$ as desired. □

Proposition 2.3.10. *For every* $\varepsilon \in 1 + \mathfrak{m}_F^*$*, there exists some* $y_\varepsilon \in Y$ *with*

$$\text{Div}(\log(\varepsilon)) = \sum_{n \in \mathbb{Z}} \phi^n(y_\varepsilon).$$

PROOF. Proposition 2.3.6 yields an untilt C_ε of F with $\varepsilon^{\sharp_{C_\varepsilon}} = 1$ and $(\varepsilon^{1/p})^{\sharp_{C_\varepsilon}} \neq 1$. Let $y_\varepsilon \in Y$ be the equivalence class of C_ε. Consider an arbitrary element $y \in Y$, represented by an untilt C of F. We know by Proposition 3.3.6 in Chapter II that $\ker(\log_{\mu_{p^\infty}})$ is the torsion subgroup of $1 + \mathfrak{m}_C$ where $\mathfrak{m}_C$ denotes the maximal ideal of $\mathcal{O}_C$. Since we have $\varepsilon \neq 1$ by assumption, Proposition 2.3.3 implies that $\log(\varepsilon)$ vanishes at y if and only if there exists some $n \in \mathbb{Z}$ with $(\varepsilon^{p^n})^{\sharp_C} = 1$ and $(\varepsilon^{p^{n-1}})^{\sharp_C} \neq 1$, or equivalently $(\varphi_F^n(\varepsilon))^{\sharp_C} = 1$ and $\left(\varphi_F^{n-1}(\varepsilon)\right)^{\sharp_C} \neq 1$ where φ_F denotes the Frobenius automorphism of F. Hence by Proposition 2.3.6 we deduce that $\log(\varepsilon)$ vanishes at y if and only if there exists some $n \in \mathbb{Z}$ with $y = \phi^n(y_\varepsilon)$. Since we have $\log(\varepsilon) \in B^{\varphi=p}$, the assertion follows by Proposition 2.3.5 and Proposition 2.3.9. □

Proposition 2.3.11. *There exists a natural bijection* $(1 + \mathfrak{m}_F^*)/\mathbb{Q}_p^\times \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$ *which maps the* $\mathbb{Q}_p^\times$*-orbit of an element* $\varepsilon \in 1 + \mathfrak{m}_F^*$ *to the set of elements in* Y *at which* $\log(\varepsilon)$ *vanishes.*

PROOF. Lemma 2.3.8 implies that the Frobenius action ϕ on Y corresponds to the multiplication by $1/p$ on $(1+\mathfrak{m}_F^*)/\mathbb{Z}_p^\times$ under the bijection $Y \xrightarrow{\sim} (1+\mathfrak{m}_F^*)/\mathbb{Z}_p^\times$ given by Proposition 2.3.6. Hence we obtain a natural bijection $(1 + \mathfrak{m}_F^*)/\mathbb{Q}_p^\times \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$. Let us now consider an arbitrary element $\varepsilon \in 1 + \mathfrak{m}_F^*$. Its $\mathbb{Q}_p^\times$-orbit maps to the ϕ-orbit of an element $y \in Y$ with a representative C that satisfies $\varepsilon^\sharp = 1$. Then we find $\widehat{\theta_C}(\log(\varepsilon)) = \log_{\mu_{p^\infty}}(\varepsilon^\sharp) = 0$ by Proposition 2.3.3, and consequently deduce the desired assertion by Proposition 2.3.10. □

2.4. Points and regularity

In this subsection, we prove that the Fargues-Fontaine curve is a Dedekind scheme whose closed points classify the Frobenius orbits in Y. For the rest of this chapter, let us write $P := \bigoplus B^{\varphi=p^n}$ and denote by $|X|$ the set of closed points in X. We also invoke the following technical result without proof.

Proposition 2.4.1. *Let f and g be elements in B. Then f is divisible by g in B if and only if we have $\text{ord}_y(f) \geq \text{ord}_y(g)$ for all $y \in Y$.*

Remark. This is one of the most difficult results from the original work of Fargues and Fontaine [**FF18**]. Curious readers can find a complete proof in [**Lur**, Lecture 13-16]. Here we provide a brief sketch of the proof.

We only need to prove the if part as the converse is obvious by Lemma 2.2.10. Moreover, in light of Lemma 2.2.2 we may replace B by $B_{[a,b]}$ for an arbitrary interval $[a, b] \subseteq (0, 1)$. The key point is to show that every element in $B_{[a,b]}$ admits a (necessarily unique) factorization into primitive elements. By a similar argument as in Proposition 2.2.11 the proof boils down to showing that every $h \in B_{[a,b]}$ with $\partial_- \mathcal{L}_{h,[a,b]}(s) \neq \partial_+ \mathcal{L}_{h,[a,b]}(s)$ for some $s \in [-\log_p(b), -\log_p(a)]$ has a zero $y \in Y_{[p^{-s},p^{-s}]}$.

Let us set $\widehat{Y} := Y \cup \{ o \}$, where o denotes the equivalence class of F as the trivial untilt of itself. Then $\widehat{Y}$ turns out to be complete with respect to an ultrametric d given by

$$d(y_1, y_2) := |\theta_{C_2}(\xi_1)|_{C_2} \qquad \text{for all } y_1, y_2 \in \widehat{Y}$$

where ξ_1 and C_2 respectively denote a primitive element that vanishes at y_1 and an untilt of F that represents y_2. If h is an element in $A_{\text{inf}}[1/p, 1/[\varpi]]$, an elegant approximation argument using Legendre-Newton polygons allows us to construct a zero $y \in Y_{[p^{-s},p^{-s}]}$ of h as the limit of a Cauchy sequence (y_n) in $\widehat{Y}$ with $|y_n| = p^{-s}$ and $\lim\limits_{n\to\infty} |h(y_n)|_{C_n} = 0$ where each C_n is a representative of y_n. For the general case, we can construct Cauchy sequences (h_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ and (y_n) in $Y_{[p^{-s},p^{-s}]}$ with $h_n(y_n) = 0$ and $\lim\limits_{n\to\infty} h_n = h$ with respect to the Gauss p^{-s}-norm, thereby obtaining a zero $y \in Y_{[p^{-s},p^{-s}]}$ of h as the limit of (y_n).

Corollary 2.4.2. *The ring $B^{\varphi=1}$ is a field.*

Proof. Consider an arbitrary nonzero element $f \in B^{\varphi=1}$. We have $\text{Div}(f) = 0$, since otherwise f would be divisible by some $g \in B^{\varphi=1/p}$, thereby contradicting Proposition 2.1.15. Hence by Proposition 2.4.1 we deduce that f admits an inverse in $B^{\varphi=1}$ as desired. □

Remark. As remarked after Proposition 2.1.15, we will see in Proposition 3.1.6 that $B^{\varphi=1}$ is canonically isomorphic to $\mathbb{Q}_p$.

Lemma 2.4.3. *Let f be an element in $B^{\varphi=p^n}$ for some $n \geq 0$, and let ε be an element in $1 + \mathfrak{m}_F^*$. Assume that both f and $\log(\varepsilon)$ vanish at some $y \in Y$. Then there exists some $g \in B^{\varphi=p^{n-1}}$ with $f = \log(\varepsilon)g$.*

Proof. By Proposition 2.3.9 we have

$$\text{ord}_{\phi^i(y)}(f) = \text{ord}_y(f) \geq 1 \qquad \text{for all } i \in \mathbb{Z}.$$

In addition, by Proposition 2.3.10 we find

$$\text{Div}(\log(\varepsilon)) = \sum_{i \in \mathbb{Z}} \phi^i(y).$$

Since $\log(\varepsilon)$ belongs to $B^{\varphi=p}$ by construction, the assertion follows by Proposition 2.4.1. □

Proposition 2.4.4. *For every $\varepsilon \in 1 + \mathfrak{m}_F$, the element $\log(\varepsilon) \in B^{\varphi=p}$ is a prime in P.*

Proof. The assertion is obvious for $\varepsilon = 1$ as P is an integral domain by Corollary 2.1.17. We henceforth assume $\varepsilon \neq 1$. Consider arbitrary elements f and g in P such that $\log(\varepsilon)$ divides fg in P. We wish to show that $\log(\varepsilon)$ divides either f or g in P. Since $\log(\varepsilon)$ is homogeneous, we may assume without loss of generality that both f and g are homogeneous. Proposition 2.3.5 implies that $\log(\varepsilon)$ vanishes at some $y_\varepsilon \in Y$. Then we find by Lemma 2.2.10 that either f or g vanishes at y_ε, and in turn deduce the desired assertion by Lemma 2.4.3. □

Proposition 2.4.5. *Let f be a nonzero element in $B^{\varphi=p^n}$ for some $n \geq 0$.*

(1) The map φ uniquely extends to an automorphism $\varphi[1/f]$ on $B[1/f]$.
(2) We may write

$$f = \lambda \log(\varepsilon_1) \cdots \log(\varepsilon_n) \qquad \text{with } \lambda \in B^{\varphi=1} \text{ and } \varepsilon_i \in 1 + \mathfrak{m}_F^* \tag{2.10}$$

where the factors are uniquely determined up to $\mathbb{Q}_p^\times$-multiple.

Proof. The first statement is straightforward to verify. Let us prove the second statement by induction on n. Since the assertion is obvious for $n = 0$, we henceforth assume $n > 0$. Then f vanishes at some $y \in Y$; otherwise, it would be invertible in B by Proposition 2.4.1 and thus would yield a nonzero element $f^{-1} \in B^{\varphi=p^{-n}}$, contradicting Proposition 2.1.15. Now Lemma 2.4.3 and Proposition 2.3.11 together yield some $\varepsilon_n \in 1 + \mathfrak{m}_F$ and $g \in B^{\varphi=p^{n-1}}$ with $f = \log(\varepsilon_n)g$. Hence by induction hypothesis we obtain an expression as in (2.10), where the factors are uniquely determined up to $\mathbb{Q}_p^\times$-multiple by Proposition 2.4.4. □

Definition 2.4.6. Given a nonzero homogeneous element $f \in P$, we refer to the map $\varphi[1/f]$ described in Proposition 2.4.5 as the *Frobenius automorphism* of $B[1/f]$. We often abuse notation and write φ instead of $\varphi[1/f]$.

Proposition 2.4.7. *Every non-generic point $x \in X$ is a closed point, induced by a prime $\log(\varepsilon)$ in P for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. Moreover, its residue field is naturally isomorphic to the perfectoid field given by any $y \in Y$ at which $\log(\varepsilon)$ vanishes.*

Proof. By Proposition 2.4.5 there exists a nonzero element $t \in B^{\varphi=p}$ such that x lies in the open subscheme $\operatorname{Spec}(B[1/t]^{\varphi=1})$ of $X = \operatorname{Proj}(P)$. Let us denote by $\mathfrak{p}$ the prime ideal of $B[1/t]^{\varphi=1}$ which corresponds to x, and take an element $f/t^n \in \mathfrak{p}$ with $f \in B^{\varphi=p^n}$. By Proposition 2.4.5 we may write

$$\frac{f}{t^n} = \lambda \cdot \frac{\log(\varepsilon_1)}{t} \cdot \frac{\log(\varepsilon_2)}{t} \cdots \frac{\log(\varepsilon_n)}{t} \qquad \text{with } \lambda \in B^{\varphi=1} \text{ and } \varepsilon_i \in 1 + \mathfrak{m}_F^*.$$

Since λ is a unit in $B^{\varphi=1}$ by Corollary 2.4.2, we have $\log(\varepsilon)/t \in \mathfrak{p}$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$.

Take an element $y \in Y$ at which $\log(\varepsilon)$ vanishes, and choose a representative C of y. Then t does not vanish at y, since otherwise Corollary 2.4.2 and Lemma 2.4.3 together would imply that $\log(\varepsilon)/t$ is an invertible element in $B^{\varphi=1}$, which is impossible as $\mathfrak{p}$ is a prime ideal. We thus obtain a map $\theta_x : B[1/t]^{\varphi=1} \hookrightarrow B[1/t] \twoheadrightarrow C$ where the second arrow is induced by $\widehat{\theta_C}$.

It suffices to show that θ_x is a surjective map whose kernel is generated by $\log(\varepsilon)/t$. Proposition 2.3.3 implies that $\widehat{\theta_C}$ induces a surjection $B^{\varphi=p} \twoheadrightarrow C$, which in turn implies that θ_x is already surjective when restricted to $(1/t)B^{\varphi=p}$. Let us now consider an arbitrary element $f'/t^n \in \ker(\theta_x)$ with $f' \in B^{\varphi=p^n}$. Arguing as in the first paragraph, we find that f'/t^n is divisible by $\log(\varepsilon')/t \in \ker(\theta_x)$ for some $\varepsilon' \in 1 + \mathfrak{m}_F^*$. Then we have $\widehat{\theta_C}(\log(\varepsilon')) = 0$, which means that $\log(\varepsilon')$ vanishes at y. Therefore we deduce by Lemma 2.4.3 that $\log(\varepsilon)/t$ divides $\log(\varepsilon')/t$, and thus divides f'/t as desired. □

Theorem 2.4.8 (Fargues-Fontaine [**FF18**]). *The scheme X has the following properties:*

(i) There exists a natural bijection $|X| \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$ which maps the point induced by $\log(\varepsilon)$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$ to the set of elements in Y at which $\log(\varepsilon)$ vanishes.

(ii) X is a Dedekind scheme such that the open subscheme $X\setminus\{\,x\,\}$ for every $x \in |X|$ is the spectrum of a principal ideal domain.

(iii) For every $x \in |X|$, its completed local ring $\widehat{\mathcal{O}_{X,x}}$ admits a natural identification

$$\widehat{\mathcal{O}_{X,x}} \cong B_{\mathrm{dR}}^+(y)$$

where y is any element in the image of x under the bijection $|X| \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$.

Proof. Proposition 2.4.7 yields a surjective map $1 + \mathfrak{m}_F^* \twoheadrightarrow |X|$ which associates to each $\varepsilon \in 1 + \mathfrak{m}_F^*$ the point $x \in X$ induced by the prime $\log(\varepsilon) \in P$. Moreover, Lemma 2.4.3 implies that two elements ε_1 and ε_2 in $1 + \mathfrak{m}_F^*$ map to the same point in $|X|$ if and only if $\log(\varepsilon_1)$ and $\log(\varepsilon_2)$ have a common zero. Therefore we deduce the property (i) by Proposition 2.3.11.

Let us now fix a closed point x in X. As shown in the preceding paragraph, the point x is induced by $\log(\varepsilon)$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. It follows that $X\setminus\{\,x\,\}$ is the spectrum of the ring $B[1/\log(\varepsilon)]^{\varphi=1}$. In addition, we find by Proposition 2.4.7 that every prime ideal of $B[1/\log(\varepsilon)]^{\varphi=1}$ is a principal ideal. Therefore we obtain the property (ii) by a general fact as stated in [**Sta**, Tag 05KH].

It remains to establish the property (iii). Let us fix an element $y \in Y$ at which $\log(\varepsilon)$ vanishes, and take an untilt C of F which represents y. We also choose an element $t \in B^{\varphi=p}$ which is not divisible by $\log(\varepsilon)$. Then we have a surjective map $\widehat{\theta_C}[1/t] : B[1/t] \twoheadrightarrow C$ induced by $\widehat{\theta_C}$. Let us denote by θ_x the restriction of $\widehat{\theta_C}[1/t]$ to $B[1/t]^{\varphi=1}$. Proposition 2.4.7 implies that we may identify x as a point in $\mathrm{Spec}\,(B[1/t]^{\varphi=1})$ given by $\ker(\theta_x)$. Hence we obtain an identification

$$\widehat{\mathcal{O}_{X,x}} \cong \varprojlim_j B[1/t]^{\varphi=1}/\ker(\theta_x)^j. \tag{2.11}$$

Meanwhile, Proposition 2.2.7 allows us to identify $B_{\mathrm{dR}}^+(y)$ as the completed local ring of a closed point $\widehat{y} \in \mathrm{Spec}\,(B)$ given by $\ker(\widehat{\theta_C})$, thereby yielding an identification

$$B_{\mathrm{dR}}^+(y) \cong \varprojlim_j B[1/t]/\ker(\widehat{\theta_C}[1/t])^j. \tag{2.12}$$

For an arbitrary element $f/t^n \in B[1/t]^{\varphi=1} \cap \ker(\widehat{\theta_C})^j$ with $f \in B^{\varphi=p^n}$ and $j \geq 1$, we have $\mathrm{ord}_y(f) \geq j$ and consequently find by Lemma 2.4.3 that f/t^n is divisible by $\log(\varepsilon)^j/t^j$. Since $\log(\varepsilon)/t$ belongs to $\ker(\theta_x)$, we obtain an identification

$$B[1/t]^{\varphi=1} \cap \ker(\widehat{\theta_C})^j = \ker(\theta_x)^j \qquad \text{for all } j \geq 1$$

and in turn get a natural injective map

$$\varprojlim_j B[1/t]^{\varphi=1}/\ker(\theta_x)^j \hookrightarrow \varprojlim_j B[1/t]/\ker(\widehat{\theta_C}[1/t])^j. \tag{2.13}$$

Moreover, since both $B[1/t]^{\varphi=1}/\ker(\theta_x)$ and $B[1/t]/\ker(\widehat{\theta_C}[1/t])$ are isomorphic to C, the map (2.13) is surjective by a general fact as stated in [**Sta**, Tag 0315]. Therefore we obtain the property (iii) by (2.11) and (2.12). □

Remark. The scheme X is defined over $\mathbb{Q}_p$ as we will see in Corollary 3.1.7. However, it is not of finite type over $\mathbb{Q}_p$ since the residue field of an arbitrary closed point is an infinite extension of $\mathbb{Q}_p$ by Proposition 2.4.7.

3. Vector bundles

Our main objective in this section is to discuss several key properties of vector bundles on the Fargues-Fontaine curve. The primary references for this section are Fargues and Fontaine's survey paper [**FF14**] and Lurie's notes [**Lur**].

3.1. Frobenius eigenspaces

In order to study the vector bundles on X, it is crucial to understand the structure of the graded ring $P = \bigoplus B^{\varphi=p^n}$. In this subsection, we aim to establish an explicit description of the Frobenius eigenspaces $B^{\varphi=p^n}$ for all $n \geq 0$.

Proposition 3.1.1. *The natural map $F \longrightarrow B$ given by Teichmüller lifts is continuous.*

Proof. Take a characteristic 0 untilt C of F. The natural map $F \longrightarrow B$ composed with $\widehat{\theta_C}$ coincides with the sharp map associated to C, which is evidently continuous by construction. Hence the assertion follows by Proposition 1.2.16. □

Lemma 3.1.2. *For every $f \in B$ with $|f|_\rho \leq 1$ for all $\rho \in (0,1)$, there exists a sequence (f_n) in $A_{\text{inf}}[1/[\varpi]]$ which converges to f with respect to all Gauss norms.*

Proof. We may assume $f \neq 0$, since the assertion is obvious for $f = 0$. Take a sequence $(\widetilde{f_n})$ in $A_{\text{inf}}[1/p, 1/[\varpi]]$ which converges to f with respect to all Gauss norms. For each $n \geq 1$, we may write $\widetilde{f_n} = f_n + \sum_{i<0} [c_{n,i}]p^i$ with $c_{n,i} \in F$ and $f_n \in A_{\text{inf}}[1/[\varpi]]$. Take arbitrary real numbers $\rho \in (0,1)$ and $\epsilon > 0$. Then for all sufficiently large n we have

$$\left|\widetilde{f_n} - f_n\right|_\rho = \sup_{i<0}\left(|c_{n,i}|\,\rho^i\right) \leq \sup_{i<0}\left(\epsilon^{-i}\right) \cdot \sup_{i<0}\left(|c_{n,i}|\,\epsilon^i\rho^i\right) \leq \epsilon \cdot \left|\widetilde{f_n}\right|_{\epsilon\rho} = \epsilon\,|f|_{\epsilon\rho} \leq \epsilon$$

where the second identity follows from Lemma 2.1.8. Hence we obtain $\lim_{n\to\infty}\left|\widetilde{f_n} - f_n\right|_\rho = 0$ for all $\rho \in (0,1)$, thereby deducing that (f_n) converges to f with respect to all Gauss norms. □

Proposition 3.1.3. *Let f be an element in B. Assume that there exists an integer $n \geq 0$ with $|f|_\rho \leq \rho^n$ for all $\rho \in (0,1)$. Then we may write $f = [c]p^n + g$ for some $c \in \mathcal{O}_F$ and $g \in B$ with $|g|_\rho \leq \rho^{n+1}$ for all $\rho \in (0,1)$.*

Proof. We may replace f by f/p^n to assume $n = 0$. Lemma 3.1.2 yields a sequence (f_i) in $A_{\text{inf}}[1/[\varpi]]$ which converges to f with respect to all Gauss norms. For each $i \geq 1$, we denote by $[c_i]$ the first coefficient in the Teichmüller expansion of f_i. Then we have $|c_{i+1} - c_i| \leq |f_{i+1} - f_i|_\rho$ for all $i \geq 1$ and $\rho \in (0,1)$. This means that the sequence (c_i) is Cauchy in F and thus converges to an element $c \in F$. In addition, given a real number $\rho \in (0,1)$, Lemma 2.1.8 yields $|c_i| \leq |f_i|_\rho = |f|_\rho \leq 1$ for all sufficiently large i, thereby implying $c \in \mathcal{O}_F$.

Let us now set $g_i := f_i - [c_i] \in A_{\text{inf}}[1/[\varpi]]$ for each $i \geq 1$ and take $g := f - [c] \in B$. We may assume $g \neq 0$, since the assertion is obvious if we have $g = 0$. Each g_i admits a Teichmüller expansion where only positive powers of p occur, so that all slopes of $\mathcal{L}_{g_i}$ are positive integers by Proposition 2.1.7. Moreover, Proposition 3.1.1 implies that the sequence (g_i) converges to g with respect to all Gauss norms. Therefore we deduce by Lemma 2.1.8 that all slopes of $\mathcal{L}_g$ are positive integers. We then use Lemma 2.1.2 to obtain

$$\mathcal{L}_g(s) \geq \min\left(\mathcal{L}_f(s), \mathcal{L}_{[c]}(s)\right) = \min\left(-\log_p\left(|f|_{p^{-s}}\right), -\log_p\left(|c|\right)\right) \geq 0 \qquad \text{for all } s > 0,$$

thereby deducing $\mathcal{L}_g(s) \geq s$ for all $s > 0$, or equivalently $|g|_\rho \leq \rho$ for all $\rho \in (0,1)$. □

Proposition 3.1.4. *Let f be a nonzero element in B.*

(1) *The element f belongs to A_{inf} if and only if we have $|f|_\rho \leq 1$ for all $\rho \in (0,1)$.*
(2) *The element f belongs to $A_{\text{inf}}[1/p]$ if and only if there exists an integer n with $|f|_\rho \leq \rho^n$ for all $\rho \in (0,1)$.*
(3) *The element f belongs to $A_{\text{inf}}[1/[\varpi]]$ if and only if there exists a constant $C > 0$ with $|f|_\rho \leq C$ for all $\rho \in (0,1)$.*
(4) *The element f belongs to $A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exist a constant $C > 0$ and an integer n with $|f|_\rho \leq C\rho^n$ for all $\rho \in (0,1)$.*

Proof. If f belongs to A_{inf}, then we clearly have $|f|_\rho \leq 1$ for all $\rho \in (0,1)$. Conversely, if we have $|f|_\rho \leq 1$ for all $\rho \in (0,1)$, then by Proposition 3.1.3 we can inductively construct a sequence (c_i) in $\mathcal{O}_F$ with

$$\left| f - \sum_{i=0}^{n-1} [c_i] p^i \right|_\rho \leq \rho^n \qquad \text{for all } n \geq 0 \text{ and } \rho \in (0,1),$$

thereby deducing $f \in A_{\text{inf}}$. Therefore we establish the statement (1).

Now we find that f belongs to $A_{\text{inf}}[1/p]$ if and only if there exists an integer n with $p^n f \in A_{\text{inf}}$, or equivalently $|f|_\rho \leq |p|_\rho^{-n} = \rho^{-n}$ for all $\rho \in (0,1)$, thereby obtaining the statement (2). Similarly, we find that f belongs to $A_{\text{inf}}[1/[\varpi]]$ if and only if there exists an integer n with $[\varpi^n] f \in A_{\text{inf}}$, or equivalently $|f|_\rho \leq |[\varpi]|_\rho^{-n} = |\varpi|^{-n}$ for all $\rho \in (0,1)$, thereby obtaining the statement (3). Finally, we find that f belongs to $A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exist integers l and n with $p^n [\varpi]^l f \in A_{\text{inf}}$, or equivalently $|f|_\rho \leq \left|[\varpi]^l p^n\right|_\rho = |\varpi|^l \rho^n$ for all $\rho \in (0,1)$, thereby obtaining the statement (4). $\square$

Lemma 3.1.5. *Given a nonzero element $f \in B^{\varphi=1}$, there exists an integer n with $|f|_\rho = \rho^n$ for all $\rho \in (0,1)$.*

Proof. By Lemma 2.1.14 we have

$$p\mathcal{L}_f(s) = \mathcal{L}_{\varphi(f)}(ps) = \mathcal{L}_f(ps) \qquad \text{for all } s > 0, \tag{3.1}$$

and consequently find $p\partial_+\mathcal{L}_f(s) = p\partial_+\mathcal{L}_f(ps)$ for all $s > 0$. Hence Corollary 2.1.11 implies that $\mathcal{L}_f$ is linear with integer slope, which means that there exist an integer n and a real number r with $\mathcal{L}_f(s) = ns + r$ for all $s > 0$. We then find $r = 0$ by (3.1), and in turn obtain $\mathcal{L}_f(s) = ns$ for all $s > 0$, or equivalently $|f|_\rho = \rho^n$ for all $\rho \in (0,1)$. $\square$

Proposition 3.1.6. *The ring $B^{\varphi=1}$ is canonically isomorphic to $\mathbb{Q}_p$.*

Proof. Let $W(\mathbb{F}_p)$ denote the ring of Witt vectors over $\mathbb{F}_p$. Under the identification

$$\mathbb{Q}_p \cong W(\mathbb{F}_p)[1/p] \cong \left\{ \sum [c_n] p^n \in A_{\text{inf}}[1/p] : c_n \in \mathbb{F}_p \right\}, \tag{3.2}$$

we may regard $\mathbb{Q}_p$ as a subring of $B^{\varphi=1}$. Let us now consider an arbitrary nonzero element $f \in B^{\varphi=1}$. Proposition 3.1.4 and Lemma 3.1.5 together imply that f is an element in $A_{\text{inf}}[1/p]$. Hence we may write $f = \sum [c_n] p^n$ with $c_n \in \mathcal{O}_F$. Since f is invariant under φ, for each $n \in \mathbb{Z}$ we find $c_n^p = c_n$, or equivalently $c_n \in \mathbb{F}_p$. We thus deduce $f \in \mathbb{Q}_p$ under the identification (3.2), thereby completing the proof. $\square$

Remark. Our proof does not depend on Proposition 2.4.1 that we assume without proof.

Corollary 3.1.7. *The scheme X is defined over $\mathbb{Q}_p$.*

Proposition 3.1.8. *The map* $\log : 1+\mathfrak{m}_F \longrightarrow B^{\varphi=p}$ *is a continuous* $\mathbb{Q}_p$*-linear isomorphism.*

Proof. Choose a characteristic 0 untilt C of F. The sharp map associated to C is continuous by construction. In addition, the map $\log_{\mu_{p^\infty}}$ is continuous by Proposition 3.3.6 in Chapter II. Therefore it follows by Proposition 2.3.3 and Proposition 1.2.16 that the map log is continuous. Moreover, since every element in $\mathbb{Q}_p$ is the limit of a sequence in $\mathbb{Z}$, we obtain the $\mathbb{Q}_p$-linearity of log by Proposition 2.3.1, and consequently deduce the surjectivity of log by Proposition 2.4.5 and Proposition 3.1.6. We also find that log is injective, as Proposition 2.3.10 yields $\log(\varepsilon) \neq 0$ for every $\varepsilon \in 1+\mathfrak{m}_F^*$. Therefore we establish the desired assertion. $\square$

Corollary 3.1.9. *There exists a natural bijection* $|X| \xrightarrow{\sim} (B^{\varphi=p}\setminus \{ 0 \})/\mathbb{Q}_p^\times$ *which maps the point induced by* $\log(\varepsilon)$ *for some* $\varepsilon \in 1+\mathfrak{m}_F^*$ *to the* $\mathbb{Q}_p^\times$*-orbit of* $\log(\varepsilon)$ *in* $B^{\varphi=p}$*.*

Proof. This is merely a restatement of the property (i) in Theorem 2.4.8 using Proposition 3.1.8. $\square$

Corollary 3.1.10. *Let* f *be a nonzero element in* $B^{\varphi=p^n}$ *for some* $n \geq 1$*. We may write*

$$f = \log(\varepsilon_1)\log(\varepsilon_2)\cdots\log(\varepsilon_n) \qquad \text{with } \varepsilon_i \in 1+\mathfrak{m}_F^*$$

where the factors are uniquely determined up to $\mathbb{Q}_p^\times$*-multiple.*

Proof. This is an immediate consequence of Proposition 3.1.6, Proposition 3.1.8, and Proposition 2.4.5. $\square$

Remark. Corollary 3.1.9 and Corollary 3.1.10 are respectively analogues of the following facts about the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}\,(\mathbb{C}[z_1, z_2])$:

(1) Closed points in $\mathbb{P}^1_{\mathbb{C}}$ are in bijection with the $\mathbb{Q}_p$-orbits of linear homogeneous polynomials in $\mathbb{C}[z_1, z_2]$.
(2) Every homogeneous polynomial in $\mathbb{C}[z_1, z_2]$ of positive degree admits a unique factorization into linear homogeneous polynomials up to $\mathbb{C}^\times$-multiple

It is therefore reasonable to expect that the Fargues-Fontaine curve X is geometrically similar to $\mathbb{P}^1_{\mathbb{C}}$, even though X is not of finite type over $\mathbb{Q}_p$. We will solidify this idea in the next subsection by studying line bundles on the Fargues-Fontaine curve.

Proposition 3.1.11. *Let* B^+ *be the closure of* $A_{\mathrm{inf}}[1/p]$ *in* B*. For every* $n \in \mathbb{Z}$ *we have* $B^{\varphi=p^n} \subseteq B^+$*.*

Proof. For $n \leq 0$, the assertion is obvious by Proposition 2.1.15 and Proposition 3.1.6. Moreover, we find

$$\log(\varepsilon) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \in B^+ \qquad \text{for every } \varepsilon \in 1+\mathfrak{m}_F$$

as each summand belongs to $A_{\mathrm{inf}}[1/p]$, thereby deducing the assertion for $n \geq 1$ by Corollary 3.1.10. $\square$

Remark. For every nonzero element $f \in B^{\varphi=n}$, we find $\lim\limits_{s\to 0}\mathcal{L}_f(s) = 0$ by the functional equation $p\mathcal{L}_f(s) = ns + \mathcal{L}_f(ps)$ as obtained in the proof of Proposition 2.1.15. Hence we can alternatively deduce Proposition 3.1.11 from an identification

$$B^+ = \left\{ f \in B : \lim_{s\to 0}\mathcal{L}_f(s) \geq 0 \right\}$$

which is not hard to verify using Proposition 2.1.9 and Proposition 3.1.4. We note that this proof does not rely on Proposition 2.4.1 which we assume without proof.

3.2. Line bundles and their cohomology

In this subsection, we classify and study line bundles on the Fargues-Fontaine curve. Throughout this subsection, we denote by $\mathrm{Div}(X)$ the group of Weil divisors on X, and by $\mathrm{Pic}(X)$ the Picard group of X. In addition, for every rational section f on X we write $\mathrm{Div}(f)$ for its associated Weil divisor on X.

Definition 3.2.1. We define the *divisor degree map* of X to be the group homomorphism $\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}$ with $\deg(x) = 1$ for all $x \in |X|$.

Proposition 3.2.2. *For every* $D \in \mathrm{Div}(X)$*, we have* $\deg(D) = 0$ *if and only if* D *is principal.*

PROOF. Let $K(X)$ denote the function field of X. We also let Q denote the fraction field of P. Note that there exists a natural identification

$$K(X) \cong \left\{ f/g \in Q : f, g \in B^{\varphi = p^n} \text{ for some } n \geq 0 \right\}. \tag{3.3}$$

Consider an arbitrary element $f \in K(X)^\times$. By (3.3) and Corollary 3.1.10 there exist some nonzero elements $t_1, t_2, \cdots, t_{2n} \in B^{\varphi = p}$ with

$$f = \frac{t_1 t_2 \cdots t_n}{t_{n+1} t_{n+2} \cdots t_{2n}}.$$

We then find $\deg(\mathrm{Div}(f)) = 0$ as Corollary 3.1.9 yields $x_1, x_2, \cdots, x_{2n} \in |X|$ with $\mathrm{Div}(t_i) = x_i$.

Let us now consider an arbitrary Weil divisor D on X with $\deg(D) = 0$. We may write

$$D = (x_1 + x_2 + \cdots + x_n) - (x_{n+1} + x_{n+2} + \cdots + x_{2n}) \qquad \text{with } x_i \in |X|.$$

Moreover, Corollary 3.1.9 yields $t_1, t_2, \cdots, t_{2n} \in B^{\varphi = p}$ with $\mathrm{Div}(t_i) = x_i$. Hence we have

$$D = \mathrm{Div}\left(\frac{t_1 t_2 \cdots t_n}{t_{n+1} t_{n+2} \cdots t_{2n}} \right),$$

which is easily seen to be a principal divisor by (3.3). □

Definition 3.2.3. For every $d \in \mathbb{Z}$, we write $P(d) := \bigoplus_{n \in \mathbb{Z}} B^{\varphi = p^{d+n}}$ and define the *d-th twist of* $\mathcal{O}_X$ to be the quasicoherent sheaf $\mathcal{O}(d)$ on X associated to $P(d)$.

Lemma 3.2.4. *For every* $d \in \mathbb{Z}$*, the sheaf* $\mathcal{O}(d)$ *is a line bundle on* X *with a canonical isomorphism* $\mathcal{O}(d) \cong \mathcal{O}(1)^{\otimes d}$*.*

PROOF. The assertion follows from Corollary 3.1.10 by a general fact as stated in [**Sta**, Tag 01MT]. □

Proposition 3.2.5. *The divisor degree map of* X *induces a natural isomorphism* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *whose inverse maps each* $d \in \mathbb{Z}$ *to the isomorphism class of* $\mathcal{O}(d)$*.*

PROOF. Since X is a Dedekind scheme as noted in Theorem 2.4.8, we can identify $\mathrm{Pic}(X)$ with the class group of X. Hence by Proposition 3.2.2 the divisor degree map of X induces a natural isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$. Let us now choose a nonzero element $t \in B^{\varphi = p}$, which induces a closed point x on X by Corollary 3.1.9. It is straightforward to check that t is a global section of $\mathcal{O}(1)$, which in turn implies by Lemma 3.2.4 that $\mathcal{O}(1)$ is isomorphic to the line bundle that arises from the Weil divisor $\mathrm{Div}(t) = x$ on X. Hence the isomorphism class of $\mathcal{O}(1)$ maps to $\deg(x) = 1$ under the isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$. The assertion now follows by Lemma 3.2.4. □

Remark. Proposition 3.2.5 is an analogue of the fact that there exists a natural isomorphism $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$ whose inverse maps each $d \in \mathbb{Z}$ to the isomorphism class of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d)$.

Proposition 3.2.6. *Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded P-module, and let $\widetilde{M}$ be the associated quasicoherent $\mathcal{O}_X$-module. There exists a canonical functorial $\mathbb{Q}_p$-linear map $M_0 \longrightarrow H^0(X, \widetilde{M})$.*

PROOF. Since we have $B^{\varphi=1} \cong \mathbb{Q}_p$ as noted in Proposition 3.1.6, the assertion follows by a general fact as stated in [**Sta**, Tag 01M7]. □

Definition 3.2.7. Given a graded P-module M, we refer to the map $M_0 \longrightarrow H^0(X, \widetilde{M})$ in Proposition 3.2.6 as the *saturation map* for M.

Proposition 3.2.8. *Let d be a nonnegative integer, and let t be a nonzero element in $B^{\varphi=p}$. The multiplication by t on P induces a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^{\varphi=p^d} & \longrightarrow & B^{\varphi=p^{d+1}} & \longrightarrow & B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & H^0(X,\mathcal{O}(d)) & \longrightarrow & H^0(X,\mathcal{O}(d+1)) & \longrightarrow & H^0(X,\mathcal{O}(d+1)/t\mathcal{O}(d)) & \longrightarrow & 0 \end{array}$$

where the vertical arrows respectively represent the saturation maps for $P(d)$, $P(d+1)$ and $P(d+1)/tP(d)$. Moreover, $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is supported at the point $x \in |X|$ induced by t.

PROOF. Since P is an integral domain by Corollary 2.1.17, the multiplication by t on P yields an exact sequence of graded P-modules

$$0 \longrightarrow P(d) \xrightarrow{f\mapsto ft} P(d+1) \longrightarrow P(d+1)/tP(d) \longrightarrow 0 \tag{3.4}$$

which gives rise to an exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}(d) \longrightarrow \mathcal{O}(d+1) \longrightarrow \mathcal{O}(d+1)/t\mathcal{O}(d) \longrightarrow 0. \tag{3.5}$$

The top row of the diagram is induced by the sequence (3.4), and is exact. The bottom row of the diagram is induced by the sequence (3.5), and is left exact. The commutativity of the diagram is evident by the functoriality of saturation maps as noted in Proposition 3.2.6.

By Corollary 3.1.8 we may write $t = \log(\varepsilon)$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. In addition, Proposition 2.3.10 yields an element $y \in Y$ at which t vanishes. Let us choose a representative C of y. Proposition 2.3.3 implies that $\widehat{\theta_C}$ restricts to a surjective map $B^{\varphi=p} \twoheadrightarrow C$. Hence for every $a \in C$ we can take $s_0, s \in B^{\varphi=p}$ with $\widehat{\theta_C}(s_0) = 1$ and $\widehat{\theta_C}(s) = a$, and consequently obtain $\widehat{\theta_C}(s_0^d s) = a$. In particular, the map $\widehat{\theta_C}$ restricts to a surjective map $B^{\varphi=p^{d+1}} \twoheadrightarrow C$. We also find by Lemma 2.4.3 that the kernel of this map is given by $tB^{\varphi=p^d}$. Therefore the map $\widehat{\theta_C}$ induces an isomorphism

$$B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} \simeq C. \tag{3.6}$$

Let us now take $x \in |X|$ induced by t. Then Proposition 2.4.7 allows us to identify C with the residue field of x. In addition, Proposition 3.2.5 implies that $\mathcal{O}(d)$ and $\mathcal{O}(d+1)$ are respectively isomorphic to the line bundles that arise from the Weil divisors dx and $(d+1)x$. It is then straightforward to verify that $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is supported at x with the stalk given by $t^{-d-1}\mathcal{O}_{X,x}/t^{-d}\mathcal{O}_{X,x} \simeq C$. This means that $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is isomorphic to the skyscraper sheaf at x with value C. Furthermore, by (3.6) we obtain an isomorphism

$$B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} \simeq C \cong H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)),$$

which is easily seen to coincide with the saturation map for $P(d+1)/tP(d)$. We then deduce by the commutativity of the second square that the bottom row is exact, thereby completing the proof. □

Theorem 3.2.9 (Fargues-Fontaine [**FF18**]). *We have the following facts about the cohomology of line bundles on X:*

(1) There exists a canonical isomorphism $H^0(X, \mathcal{O}(d)) \cong B^{\varphi=p^d}$ for every $d \in \mathbb{Z}$.
(2) The cohomology group $H^1(X, \mathcal{O}(d))$ vanishes for every $d \geq 0$.

Proof. Take a nonzero element $t \in B^{\varphi=p}$. By Corollary 3.1.9 there exists a closed point x on X induced by t. Let us write $U := X \setminus \{ x \}$. Then we have an identification $U \cong \operatorname{Spec}(B[1/t]^{\varphi=1})$.

For every $d \in \mathbb{Z}$, the multiplication by t on P yields an injective map of P-graded modules $P(d) \hookrightarrow P(d+1)$ by Corollary 2.1.17, and in turn gives rise to an injective sheaf morphism $\mathcal{O}(d) \hookrightarrow \mathcal{O}(d+1)$. In addition, Proposition 3.2.5 implies that each $\mathcal{O}(d)$ is isomorphic to the line bundle that arises from the Weil divisor dx. We then find that $\varinjlim \mathcal{O}(d)$ is naturally isomorphic to the pushforward of $\mathcal{O}_U$ by the embedding $U \hookrightarrow X$, and in turn obtain identifications

$$H^0\left(X, \varinjlim \mathcal{O}(d)\right) \cong H^0(U, \mathcal{O}_U) \cong B[1/t]^{\varphi=1}, \tag{3.7}$$

$$H^1\left(X, \varinjlim \mathcal{O}(d)\right) \cong H^1(U, \mathcal{O}_U) = 0. \tag{3.8}$$

Let us now prove the statement (1). For every $d \in \mathbb{Z}$, we denote by α_d the saturation map of $P(d)$. We wish to show that each α_d is an isomorphism. Proposition 3.2.8 implies that the sequence (α_d) gives rise to a map

$$B[1/t]^{\varphi=1} \cong \varinjlim B^{\varphi=p^d} \longrightarrow \varinjlim H^0(X, \mathcal{O}(d)) \cong H^0\left(X, \varinjlim \mathcal{O}(d)\right),$$

which is easily seen to coincide with the isomorphism (3.7). Moreover, Proposition 3.2.8 and the snake lemma together yield isomorphisms

$$\ker(\alpha_d) \simeq \ker(\alpha_{d+1}) \qquad \text{and} \qquad \operatorname{coker}(\alpha_d) \simeq \operatorname{coker}(\alpha_{d+1}) \qquad \text{for all } d \geq 0.$$

Therefore we deduce that α_d is an isomorphism for every $d \geq 0$. In particular, we have $H^0(X, \mathcal{O}_X) \cong B^{\varphi=1} \cong \mathbb{Q}_p$ where the second isomorphism is given by Proposition 3.1.6. Then for every $d < 0$, we find that there exists no nonzero element element of $H^0(X, \mathcal{O}_X)$ which vanishes to order $-d$ at x, and consequently obtain $H^0(X, \mathcal{O}(d)) = 0$. We thus deduce by Proposition 2.1.15 that α_d is an isomorphism for every $d < 0$ as well.

It remains to establish the statement (2). For every $n \geq 0$, the last statement of Proposition 3.2.8 implies that the cohomology of $\mathcal{O}(d+1)/t\mathcal{O}(d)$ vanishes in degree 1. Hence for every $d \geq 0$ we have a long exact sequence

$$H^0(X, \mathcal{O}(d+1)) \longrightarrow H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)) \longrightarrow H^1(X, \mathcal{O}(d)) \longrightarrow H^1(X, \mathcal{O}(d)) \longrightarrow 0,$$

which in turn yields an isomorphism $H^1(X, \mathcal{O}(d)) \simeq H^1(X, \mathcal{O}(d+1))$ as the first arrow is surjective by Proposition 3.2.8. The desired assertion now follows by (3.8). □

Remark. Theorem 3.2.9 provides analogues of the following facts about the complex projective line $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj}(\mathbb{C}[z_1, z_2])$:

(1) For every $d \in \mathbb{Z}$, the cohomology group $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d))$ is naturally isomorphic to the group of degree d homogeneous polynomials in $\mathbb{C}[z_1, z_2]$.
(2) For every $d \geq 0$, the cohomology group $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d))$ vanishes.

However, it is known that $H^1(X, \mathcal{O}(-1))$ does not vanish while $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1))$ vanishes.

3.3. Harder-Narasimhan filtration

In this subsection, we review the Harder-Narasimhan formalism for vector bundles on a complete algebraic curve.

Definition 3.3.1. A *complete algebraic curve* is a scheme Z with the following properties:

(i) Z is connected, separated, noetherian and regular of dimension 1.

(ii) The Picard group $\mathrm{Pic}(Z)$ admits a homomorphism $\deg_Z : \mathrm{Pic}(Z) \longrightarrow \mathbb{Z}$, called a *degree map*, which takes a positive value on every line bundle that arises from a nonzero effective Weil divisor on Z.

Example 3.3.2. Below are two important examples of complete algebraic curves.

(1) Every regular proper curve over a field is a complete algebraic curve by a general fact as stated in [**Sta**, Tag 0AYY].

(2) The Fargues-Fontaine curve is a complete algebraic curve by Theorem 2.4.8 and Proposition 3.2.5.

For the rest of this subsection, we fix a complete algebraic curve Z with a degree map $\deg_Z$ on the Picard group $\mathrm{Pic}(Z)$. Our first goal in this subsection is to study the notion of degree and slope for vector bundles on Z.

Definition 3.3.3. Let $\mathcal{V}$ be a vector bundle on Z.

(1) We write $\mathrm{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$, and define the *degree* of $\mathcal{V}$ by

$$\deg(\mathcal{V}) := \deg_Z\left(\wedge^{\mathrm{rk}(\mathcal{V})}(\mathcal{V})\right).$$

(2) If $\mathcal{V}$ is not zero, we define its *slope* by

$$\mu(\mathcal{V}) := \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}.$$

(3) We denote by $\mathcal{V}^\vee$ the dual bundle of $\mathcal{V}$.

Proposition 3.3.4. *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be vector bundles on Z. Assume that there exits a short exact sequence*

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0.$$

(1) We have identities

$$\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{U}) + \mathrm{rk}(\mathcal{W}) \qquad \text{and} \qquad \deg(\mathcal{V}) = \deg(\mathcal{U}) + \deg(\mathcal{W}).$$

(2) If $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are all nonzero, then we have

$$\min\left(\mu(\mathcal{U}), \mu(\mathcal{W})\right) \leq \mu(\mathcal{V}) \leq \max\left(\mu(\mathcal{U}), \mu(\mathcal{W})\right)$$

with equality if and only if $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$ are equal.

Proof. The first identity in the statement (1) is evident, whereas the second identity in the statement (1) follows from a general fact as stated in [**Sta**, Tag 0B38]. It remains to prove the the statement (2). Let us now assume that $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are all nonzero. By the statement (1) we have

$$\mu(\mathcal{V}) = \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})} = \frac{\deg(\mathcal{U}) + \deg(\mathcal{W})}{\mathrm{rk}(\mathcal{U}) + \mathrm{rk}(\mathcal{W})}.$$

If $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$ are not equal, then $\mu(\mathcal{V})$ must lie between $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$. Otherwise, we find $\mu(\mathcal{U}) = \mu(\mathcal{V}) = \mu(\mathcal{W})$. $\square$

Lemma 3.3.5. *Let M and N be free modules over a ring R of rank r and r'. There exists a canonical isomorphism*

$$\wedge^{rr'}(M \otimes_R N) \cong \wedge^r(M)^{\otimes r'} \otimes_R \wedge^{r'}(N)^{\otimes r}.$$

Proof. Let us choose bases (m_i) and (n_j) for M and N, respectively. We have an isomorphism of rank 1 free R-modules

$$\wedge^{rr'}(M \otimes_R N) \simeq \wedge^r(M)^{\otimes r'} \otimes_R \wedge^{r'}(N)^{\otimes r} \tag{3.9}$$

which maps $\bigwedge(m_i \otimes n_j)$ to $(\bigwedge m_i)^{\otimes r'} \otimes (\bigwedge n_j)^{\otimes r}$. It suffices to show that this map does not depend on the choices of (m_i) and (n_j). Take an invertible $r \times r$ matrix $\alpha = (\alpha_{h,i})$ over R. Then we have

$$\bigwedge\left(\sum \alpha_{h,i} m_i \otimes n_j\right) = \det(\alpha)^{r'} \bigwedge(m_i \otimes n_j),$$
$$\left(\bigwedge\left(\sum \alpha_{h,i} m_i\right)\right)^{\otimes r'} \otimes \left(\bigwedge n_j\right)^{\otimes r} = \det(\alpha)^{r'} \left(\bigwedge m_i\right)^{\otimes r'} \otimes \left(\bigwedge n_j\right)^{\otimes r}.$$

Hence $\bigwedge(\sum \alpha_{h,i} m_i \otimes n_j)$ maps to $(\bigwedge(\sum \alpha_{h,i} m_i))^{\otimes r'} \otimes (\bigwedge n_j)^{\otimes r}$ under (3.9). It follows that the map (3.9) does not depend on the choice of (m_i). By symmetry, the map (3.9) does not depend on the choice of (n_j) either. Therefore we deduce the desired assertion. □

Proposition 3.3.6. *Let $\mathcal{V}$ and $\mathcal{W}$ be nonzero vector bundles on Z. Then we have*

$$\deg(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \deg(\mathcal{V})\mathrm{rk}(\mathcal{W}) + \deg(\mathcal{W})\mathrm{rk}(\mathcal{V}) \quad \textit{and} \quad \mu(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \mu(\mathcal{V}) + \mu(\mathcal{W}).$$

Proof. Since we have $\mathrm{rk}(\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{W}) = \mathrm{rk}(\mathcal{V})\mathrm{rk}(\mathcal{W})$, the first identity is straightforward to verify by Lemma 3.3.5. The second identity then immediately follows. □

Lemma 3.3.7. *The cohomology group $H^0(Z, \mathcal{O}_Z)$ is a field.*

Proof. Let $K(Z)$ denote the function field of Z, and take an arbitrary element $f \in K(Z)^\times$. Then f yields a global section of $\mathcal{O}_Z$ if and only if the associated Weil divisor $\mathrm{Div}(f)$ on Z is effective. Since every principal divisor on Z induces a line bundle of degree 0, the Weil divisor $\mathrm{Div}(f)$ is effective if and only if it is the zero divisor. We thus find

$$H^0(Z, \mathcal{O}_Z) \backslash \{ 0 \} = \left\{ f \in K(Z)^\times : \mathrm{Div}(f) = 0 \right\},$$

and consequently deduce that $H^0(Z, \mathcal{O}_Z)$ is a subfield of $K(Z)$. □

Lemma 3.3.8. *Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on Z.*

(1) If we have $\deg(\mathcal{L}) > \deg(\mathcal{M})$, there is no nonzero $\mathcal{O}_Z$-module map from $\mathcal{L}$ to $\mathcal{M}$.
(2) If we have $\deg(\mathcal{L}) = \deg(\mathcal{M})$, every nonzero $\mathcal{O}_Z$-module map from $\mathcal{L}$ to $\mathcal{M}$ is an isomorphism.

Proof. Assume that there exists a nonzero $\mathcal{O}_Z$-module map $s : \mathcal{L} \longrightarrow \mathcal{M}$. Then s induces a nonzero global section on $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ via the identification

$$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{L}, \mathcal{M}) \cong H^0(Z, \mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}). \tag{3.10}$$

Hence $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ arises from an effective Weil divisor D on Z by a general fact as stated in [**Sta**, Tag 01X0]. We then find

$$\deg(\mathcal{M}) - \deg(\mathcal{L}) = \deg(\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}) \geq 0, \tag{3.11}$$

and consequently deduce the first statement.

Let us now assume $\deg(\mathcal{L}) = \deg(\mathcal{M})$. By (3.11) we have $\deg(\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}) = 0$, which means that the effective Weil D must be zero. It follows that $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ is trivial, which in turn implies by (3.10) and Lemma 3.3.7 that s is an isomorphism. □

Proposition 3.3.9. *A coherent $\mathcal{O}_Z$-module is a vector bundle if and only if it is torsion free.*

Proof. Since Z is integral and regular by construction, the assertion follows from a general fact as stated in [**Sta**, Tag 0CC4]. $\square$

Proposition 3.3.10. *Let $\mathcal{V}$ be a vector bundle on Z, and let $\mathcal{W}$ be a coherent subsheaf of $\mathcal{V}$.*

(1) $\mathcal{W}$ is a vector bundle on Z.

(2) $\mathcal{W}$ is contained in a subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ with $\mathrm{rk}(\mathcal{W}) = \mathrm{rk}(\widetilde{\mathcal{W}})$ and $\deg(\mathcal{W}) \leq \deg(\widetilde{\mathcal{W}})$.

Proof. Since $\mathcal{W}$ is evidently torsion free, the first statement follows from Proposition 3.3.9. Hence it remains to verify the second statement. We may assume $\mathcal{W} \neq 0$, as otherwise the assertion would be obvious. Let $\mathcal{T}$ denote the torsion subsheaf of the quotient $\mathcal{V}/\mathcal{W}$. Take $\widetilde{\mathcal{W}}$ to be the preimage of $\mathcal{T}$ under the surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{W}$. Then $\widetilde{\mathcal{W}}$ is a torsion free subsheaf of $\mathcal{V}$ with a torsion free quotient, and thus is a subbundle of $\mathcal{V}$ by Proposition 3.3.9. In addition, we have $\mathcal{W} \subseteq \widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}/\mathcal{W} \simeq \mathcal{T}$ by construction, and consequently find $\mathrm{rk}(\widetilde{\mathcal{W}}) = \mathrm{rk}(\mathcal{W})$ as $\mathcal{T}$ has rank 0 for being a torsion sheaf. We also have a nonzero $\mathcal{O}_Z$-module map $\wedge^{\mathrm{rk}(\mathcal{W})}\mathcal{W} \longrightarrow \wedge^{\mathrm{rk}(\widetilde{\mathcal{W}})}\widetilde{\mathcal{W}}$ induced by the embedding $\mathcal{W} \hookrightarrow \widetilde{\mathcal{W}}$, and in turn obtain $\deg(\mathcal{W}) \leq \deg(\widetilde{\mathcal{W}})$ by Lemma 3.3.8. $\square$

Remark. The subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ that we constructed above is often referred to as the *saturation* of $\mathcal{W}$ in $\mathcal{V}$.

Proposition 3.3.11. *Let $\mathcal{V}$ and $\mathcal{W}$ be vector bundles on Z of equal rank and degree. Assume that $\mathcal{W}$ is a coherent subsheaf of $\mathcal{V}$. Then we have $\mathcal{V} = \mathcal{W}$.*

Proof. The embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ induces a nonzero map $\wedge^{\mathrm{rk}(\mathcal{W})}(\mathcal{W}) \longrightarrow \wedge^{\mathrm{rk}(\mathcal{V})}(\mathcal{V})$, which is forced to be an isomorphism by Lemma 3.3.8. Hence at each point in Z the embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ yields an isomorphism on the stalks for having an invertible determinant. It follows that the embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ is an isomorphism. $\square$

Proposition 3.3.12. *Given a vector bundle $\mathcal{V}$ on Z, there is an integer $d_{\mathcal{V}}$ with $\deg(\mathcal{W}) \leq d_{\mathcal{V}}$ for every subbundle $\mathcal{W}$ of $\mathcal{V}$.*

Proof. If $\mathcal{V}$ is the zero bundle, the assertion is trivial. Let us now proceed by induction on $\mathrm{rk}(\mathcal{V})$. We may assume that there exists a nonzero proper subbundle $\mathcal{U}$ of $\mathcal{V}$, as otherwise the assertion would be obvious. Consider an arbitrary subbundle $\mathcal{W}$ of $\mathcal{V}$. Let us set $\mathcal{P} := \mathcal{W} \cap \mathcal{U}$ and denote by $\mathcal{Q}$ the image of $\mathcal{W}$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{U}$. Proposition 3.3.10 and the induction hypothesis together imply that $\mathcal{P}$ and $\mathcal{Q}$ are vector bundles on Z with

$$\deg(\mathcal{P}) \leq d_{\mathcal{U}} \qquad \text{and} \qquad \deg(\mathcal{Q}) \leq d_{\mathcal{V}/\mathcal{U}}$$

for some integers $d_{\mathcal{U}}$ and $d_{\mathcal{V}/\mathcal{U}}$ that do not depend on $\mathcal{W}$. In addition, we have a short exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{W} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Therefore we obtain

$$\deg(\mathcal{W}) = \deg(\mathcal{P}) + \deg(\mathcal{Q}) \leq d_{\mathcal{U}} + d_{\mathcal{V}/\mathcal{U}}$$

where the first identity follows from Proposition 3.3.4. $\square$

Remark. On the other hand, if $\mathcal{V}$ is not a line bundle on Z, we don't necessarily have an integer $d'_{\mathcal{V}}$ with $\deg(\mathcal{W}) \geq d'_{\mathcal{V}}$ for every subbundle $\mathcal{W}$ of $\mathcal{V}$. In fact, in the context of the complex projective line or the Fargues-Fontaine curve, it is known that such an integer $d'_{\mathcal{V}}$ never exists if $\mathcal{V}$ is not a line bundle.

We now introduce and study two important classes of vector bundles on Z.

Definition 3.3.13. Let $\mathcal{V}$ be a nonzero vector bundle on Z.

(1) We say that $\mathcal{V}$ is *semistable* if we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for every nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$.

(2) We say that $\mathcal{V}$ is *stable* if we have $\mu(\mathcal{W}) < \mu(\mathcal{V})$ for every nonzero proper subbundle $\mathcal{W}$ of $\mathcal{V}$.

Remark. Here we don't speak of semistability for the zero bundle, although some authors say that the zero bundle is semistable of every slope.

Example 3.3.14. Every line bundle on Z is stable; indeed, a line bundle on Z has no nonzero proper subbundles as easily seen by Proposition 3.3.4.

Proposition 3.3.15. *Let $\mathcal{V}$ be a semistable vector bundle on Z. Every nonzero coherent subsheaf $\mathcal{W}$ of $\mathcal{V}$ is a vector bundle on Z with $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$.*

PROOF. Proposition 3.3.10 implies that $\mathcal{W}$ is a vector bundle on Z, contained in some subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ with $\mu(\mathcal{W}) \leq \mu(\widetilde{\mathcal{W}})$. We then find $\mu(\widetilde{\mathcal{W}}) \leq \mu(\mathcal{V})$ by the semistability of $\mathcal{V}$, and consequently obtain the desired assertion. □

Proposition 3.3.16. *Let $\mathcal{V}$ and $\mathcal{W}$ be semistable vector bundles on Z with $\mu(\mathcal{V}) > \mu(\mathcal{W})$. Then we have $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{V}, \mathcal{W}) = 0$.*

PROOF. Suppose for contradiction that there is a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{V} \longrightarrow \mathcal{W}$. Let $\mathcal{Q}$ denote the image of f. Proposition 3.3.15 implies that $\mathcal{Q}$ is a vector bundle on Z with

$$\mu(\mathcal{Q}) \leq \mu(\mathcal{W}) < \mu(\mathcal{V}). \tag{3.12}$$

Let us now consider the short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{V} \xrightarrow{f} \mathcal{Q} \longrightarrow 0.$$

We have $\ker(f) \neq 0$ as $\mathcal{Q}$ and $\mathcal{V}$ are not isomorphic by (3.12). We thus obtain $\mu(\ker(f)) \leq \mu(\mathcal{V})$ by the semistability of $\mathcal{V}$ and consequently find $\mu(\mathcal{Q}) \geq \mu(\mathcal{V})$ by Proposition 3.3.4, thereby deducing a desired contradiction by (3.12). □

Remark. The converse of Proposition 3.3.16 does not hold in general. For example, if the Picard group of Z is not isomorphic to $\mathbb{Z}$, we get a nontrivial degree 0 line bundle $\mathcal{L}$ on Z and find $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{O}_Z, \mathcal{L}) = 0$ by Lemma 3.3.8. On the other hand, if Z is taken to be the complex projective line or the Fargues-Fontaine curve, then the converse of Proposition 3.3.16 is known to hold.

Proposition 3.3.17. *Let $\mathcal{V}$ be a vector bundle on Z such that $\mathcal{V}^{\otimes n}$ is semistable for some $n > 0$. Then $\mathcal{V}$ is semistable.*

PROOF. Consider an arbitrary nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$. We may regard $\mathcal{W}^{\otimes n}$ as a subsheaf of $\mathcal{V}^{\otimes n}$. Then we have $\mu(\mathcal{W}^{\otimes n}) \leq \mu(\mathcal{V}^{\otimes n})$ by Proposition 3.3.15, and in turn find

$$\mu(\mathcal{W}) = \mu(\mathcal{W}^{\otimes n})/n \leq \mu(\mathcal{V}^{\otimes n})/n = \mu(\mathcal{V})$$

by Proposition 3.3.6. □

Remark. It is natural to ask if the tensor product of two arbitrary semistable vector bundles on Z is necessarily semistable. If Z is a regular proper curve over a field of characteristic 0, this is known to be true by the work of Narasimhan-Seshadri [**NS65**]. In addition, we will see in Corollary 3.5.2 that this is true in the context of the Fargues-Fontaine curve. However, this is false if Z is defined over a field of characteristic p, as shown by Gieseker [**Gie73**].

Proposition 3.3.18. *Let $\mathcal{V}$ and $\mathcal{W}$ be semistable vector bundles on Z of slope λ.*

(1) *Every extension of $\mathcal{W}$ by $\mathcal{V}$ is a semistable vector bundle on Z of slope λ.*
(2) *For every $f \in \mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{V}, \mathcal{W})$, both $\ker(f)$ and $\mathrm{coker}(f)$ are either trivial or semistable vector bundles on Z of slope λ.*

Proof. Let $\mathcal{E}$ be a vector bundle on X which fits into a short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{E} \longrightarrow \mathcal{W} \longrightarrow 0.$$

By Proposition 3.3.4 we find $\mu(\mathcal{E}) = \lambda$. Take an arbitrary subbundle $\mathcal{F}$ of $\mathcal{E}$, and denote by $\mathcal{F}'$ its image under the map $\mathcal{E} \twoheadrightarrow \mathcal{W}$. By construction we have a short exact sequence

$$0 \longrightarrow \mathcal{V} \cap \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

In addition, Proposition 3.3.15 implies that $\mathcal{V} \cap \mathcal{F}$ and $\mathcal{F}'$ are vector bundles on Z with

$$\mu(\mathcal{V} \cap \mathcal{F}) \leq \mu(\mathcal{V}) = \lambda \qquad \text{and} \qquad \mu(\mathcal{F}') \leq \mu(\mathcal{W}) = \lambda.$$

We then find $\mu(\mathcal{F}) \leq \lambda = \mu(\mathcal{E})$ by Proposition 3.3.4, thereby deducing the statement (1).

It remains prove the statement (2). The assertion is trivial for $f = 0$. We henceforth assume $f \neq 0$, and denote by $\mathcal{Q}$ the image of f. Then we have a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{V} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

Moreover, Proposition 3.3.15 implies that $\ker(f)$ and $\mathcal{Q}$ are vector bundles on Z with

$$\deg(\ker(f)) \leq \mu(\mathcal{V}) \cdot \mathrm{rk}(\ker(f)) = \lambda \cdot \mathrm{rk}(\ker(f)) \qquad \text{and} \qquad \mu(\mathcal{Q}) \leq \mu(\mathcal{W}) = \lambda.$$

Hence by Proposition 3.3.4 we find

$$\deg(\ker(f)) = \lambda \cdot \mathrm{rk}(\ker(f)) \qquad \text{and} \qquad \mu(\mathcal{Q}) = \lambda.$$

Since every subbundle of $\ker(f)$ is a coherent subsheaf of $\mathcal{V}$, the first identity and Proposition 3.3.15 together imply that $\ker(f)$ is either zero or semistable of slope λ.

Meanwhile, Proposition 3.3.10 implies that $\mathcal{Q}$ is contained in a subbundle $\widetilde{\mathcal{Q}}$ of $\mathcal{W}$ with

$$\mathrm{rk}(\mathcal{Q}) = \mathrm{rk}(\widetilde{\mathcal{Q}}) \qquad \text{and} \qquad \deg(\mathcal{Q}) \leq \deg(\widetilde{\mathcal{Q}}). \tag{3.13}$$

Then by the semistability of $\mathcal{V}$ we obtain

$$\lambda = \mu(\mathcal{Q}) \leq \mu(\widetilde{\mathcal{Q}}) \leq \mu(\mathcal{W}) = \lambda,$$

and consequently find that the inequality in (3.13) is indeed an equality. Hence Proposition 3.3.11 yields $\mathcal{Q} = \widetilde{\mathcal{Q}}$, which in particular means that $\mathcal{Q}$ is a subbundle of $\mathcal{W}$.

Let us now assume that $\mathrm{coker}(f)$ is not zero. Since we have a short exact sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{W} \longrightarrow \mathrm{coker}(f) \longrightarrow 0,$$

our discussion in the preceding paragraph and Proposition 3.3.4 together imply that $\mathrm{coker}(f)$ is a vector bundle on Z with $\mu(\mathrm{coker}(f)) = \lambda$. We wish to show that $\mathrm{coker}(f)$ is semistable. Take an arbitrary subbundle $\mathcal{R}$ of $\mathrm{coker}(f)$, and denote by $\mathcal{R}'$ its preimage under the map $\mathcal{W} \twoheadrightarrow \mathrm{coker}(f)$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{R}' \longrightarrow \mathcal{R} \longrightarrow 0.$$

In addition, Proposition 3.3.15 implies that $\mathcal{R}'$ is a vector bundle on Z with

$$\mu(\mathcal{R}') \leq \mu(\mathcal{W}) = \lambda = \mu(\mathcal{Q}).$$

Hence we find $\mu(\mathcal{R}) \leq \mu(\mathcal{Q}) = \lambda = \mu(\mathrm{coker}(f))$ by Proposition 3.3.4, and consequently deduce that $\mathrm{coker}(f)$ is semistable as desired. $\square$

Our final goal in this subsection is to show that every vector bundle on Z admits a unique filtration whose successive quotients are semistable vector bundles with strictly increasing slopes.

Definition 3.3.19. Let $\mathcal{V}$ be a vector bundle on Z. A *Harder-Narasimhan filtration* of $\mathcal{V}$ is a filtration by subbundles

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$

such that the successive quotients $\mathcal{V}_1/\mathcal{V}_0, \cdots, \mathcal{V}_n/\mathcal{V}_{n-1}$ are semistable vector bundles on Z with $\mu(\mathcal{V}_1/\mathcal{V}_0) > \cdots > \mu(\mathcal{V}_n/\mathcal{V}_{n-1})$.

Lemma 3.3.20. *Given a nonzero vector bundle $\mathcal{V}$ on Z, there exists a semistable subbundle $\mathcal{V}_1$ of $\mathcal{V}$ with $\mu(\mathcal{V}_1) \geq \mu(\mathcal{V})$ and $\mu(\mathcal{V}_1) > \mu(\mathcal{U})$ for every nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$.*

Proof. For an arbitrary nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$, we have $0 < \mathrm{rk}(\mathcal{W}) \leq \mathrm{rk}(\mathcal{V})$ and $\deg(\mathcal{W}) \leq d_{\mathcal{V}}$ for some fixed integer $d_{\mathcal{V}}$ given by Proposition 3.3.12. This implies that the set

$$S := \{ q \in \mathbb{Q} : q = \mu(\mathcal{W}) \text{ for some nonzero subbundle } \mathcal{W} \text{ of } \mathcal{V} \}$$

is discrete and bounded above. In particular, the set S contains the largest element λ.

Let us take $\mathcal{V}_1$ to be a maximal subbundle of $\mathcal{V}$ with $\mu(\mathcal{V}_1) = \lambda$. By construction we have $\mu(\mathcal{V}_1) \geq \mu(\mathcal{V})$. Moreover, since every subbundle of $\mathcal{V}_1$ is a coherent subsheaf of $\mathcal{V}$, Proposition 3.3.10 and the maximality of λ together imply that $\mathcal{V}_1$ is semistable. Let us now consider an arbitrary nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$, and denote by $\widetilde{\mathcal{U}}$ its preimage under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \widetilde{\mathcal{U}} \longrightarrow \mathcal{U} \longrightarrow 0.$$

In addition, the maximality of λ and $\mathcal{V}_1$ implies $\mu(\widetilde{\mathcal{U}}) < \lambda = \mu(\mathcal{V}_1)$. Therefore we find $\mu(\mathcal{U}) < \mu(\mathcal{V}_1)$ by Proposition 3.3.4, thereby completing the proof. □

Remark. Our proof above relies on the fact that the group $\mathbb{Z}$ is discrete. However, as noted in [**Ked17**, Lemma 3.4.10], it is not hard to prove Lemma 3.3.20 without using the discreteness of $\mathbb{Z}$. As a consequence, we can extend all of our discussion in this subsection to some other contexts where the degree of a vector bundle takes a value in a nondiscrete group such as $\mathbb{Z}[1/p]$. We refer the curious readers to [**Ked17**, Example 3.5.7] for a discussion of such an example.

Lemma 3.3.21. *Let $\mathcal{V}$ be a nonzero vector bundle on Z. Assume that $\mathcal{V}$ admits a Harder-Narasimhan filtration*

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}.$$

For every semistable vector bundle $\mathcal{W}$ on Z with $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{W}, \mathcal{V}) \neq 0$, we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V}_1)$.

Proof. Take a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{W} \longrightarrow \mathcal{V}$, and denote its image by $\mathcal{Q}$. Since $\mathcal{Q}$ is a nonzero coherent subsheaf of $\mathcal{V}$, there exists the smallest integer $i \geq 1$ with $\mathcal{Q} \subseteq \mathcal{V}_i$. Then we find that f induces a nonzero $\mathcal{O}_Z$-module map $\mathcal{W} \xrightarrow{f} \mathcal{V}_i \twoheadrightarrow \mathcal{V}_i/\mathcal{V}_{i-1}$, and consequently obtain

$$\mu(\mathcal{W}) \leq \mu(\mathcal{V}_i/\mathcal{V}_{i-1}) \leq \mu(\mathcal{V}_1)$$

where the first inequality follows by Proposition 3.3.16. □

Remark. Lemma 3.3.21 does not hold without the semistability assumption on $\mathcal{W}$. For example, if we take $\mathcal{W} := \mathcal{V} \oplus \mathcal{L}$ where $\mathcal{L}$ is a line bundle on Z with $\mu(\mathcal{L}) > \mu(\mathcal{V})$, we find $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{W}, \mathcal{V}) \neq 0$ and $\mu(\mathcal{W}) > \mu(\mathcal{V})$.

Theorem 3.3.22 (Harder-Narasimhan [**HN75**]). *Every vector bundle $\mathcal{V}$ on Z admits a unique Harder-Narasimhan filtration.*

PROOF. Let us proceed by induction on $\mathrm{rk}(\mathcal{V})$. If $\mathcal{V}$ is the zero bundle, the assertion is trivial. We henceforth assume that $\mathcal{V}$ is not zero.

We first assert that $\mathcal{V}$ admits a Harder-Narasimhan filtration. Lemma 3.3.20 yields a semistable subbundle $\mathcal{V}_1$ of $\mathcal{V}$ with $\mu(\mathcal{V}_1) > \mu(\mathcal{U})$ for every nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$. By the induction hypothesis, the vector bundle $\mathcal{V}/\mathcal{V}_1$ on Z admits a Harder-Narasimhan filtration

$$0 = \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \mathcal{V}/\mathcal{V}_1. \tag{3.14}$$

For each $i = 2, \cdots, n$, let us set $\mathcal{V}_i$ to be the preimage of $\mathcal{U}_i$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$. Then we find

$$\mathcal{V}_i/\mathcal{V}_{i-1} \cong \mathcal{U}_i/\mathcal{U}_{i-1} \qquad \text{for each } i = 2, \cdots, n.$$

Moreover, by construction we have $\mu(\mathcal{V}_1) > \mu(\mathcal{U}_2)$ whenever the filtration (3.14) is not trivial. Therefore $\mathcal{V}$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}. \tag{3.15}$$

It remains to show that (3.15) is a unique Harder-Narasimhan filtration of $\mathcal{V}$. Assume that $\mathcal{V}$ admits another Harder-Narasimhan filtration

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_l = \mathcal{V}. \tag{3.16}$$

Since $\mathcal{W}_1$ is a nonzero subbundle of $\mathcal{V}$, Lemma 3.3.21 yields $\mu(\mathcal{W}_1) \leq \mu(\mathcal{V}_1)$. Then by symmetry we obtain $\mu(\mathcal{V}_1) \leq \mu(\mathcal{W}_1)$, and thus find $\mu(\mathcal{V}_1) = \mu(\mathcal{W}_1)$. Now we have

$$\mu(\mathcal{W}_1) = \mu(\mathcal{V}_1) > \mu(\mathcal{V}_2/\mathcal{V}_1) = \mu(\mathcal{U}_2/\mathcal{U}_1)$$

unless the filtration (3.14) is trivial. It follows by Lemma 3.3.21 that $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{W}_1, \mathcal{V}/\mathcal{V}_1)$ vanishes. We then find $\mathcal{W}_1 \subseteq \mathcal{V}_1$ by observing that the natural map $\mathcal{W}_1 \hookrightarrow \mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$ must be zero. By symmetry we also obtain $\mathcal{V}_1 \subseteq \mathcal{W}_1$, and consequently deduce that $\mathcal{V}_1$ and $\mathcal{W}_1$ are equal. The filtration (3.16) then induces a Harder-Narasimhan filtration

$$0 = \mathcal{W}_1/\mathcal{V}_1 \subset \cdots \mathcal{W}_l/\mathcal{V}_1 = \mathcal{V}/\mathcal{V}_1, \tag{3.17}$$

which must coincide with the filtration (3.14) by the induction hypothesis. Since each $\mathcal{W}_i$ is the preimage of $\mathcal{W}_i/\mathcal{V}_1$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$, we deduce that the filtrations (3.15) and (3.16) coincide. □

Remark. A careful examination of our proof shows that Theorem 3.3.22 is a formal consequence of Proposition 3.3.4 and Proposition 3.3.10. In other words, Theorem 3.3.22 readily extends to any exact category $\mathscr{C}$ equipped with assignments $\mathrm{rk}_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathbb{Z}_{\geq 0}$ and $\deg_{\mathscr{C}} : \mathscr{C} \longrightarrow \mathbb{Z}$ that satisfy the following properties:

(i) Both $\mathrm{rk}_{\mathscr{C}}$ and $\deg_{\mathscr{C}}$ are additive on short exact sequences.

(ii) Every monomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathscr{C}$ factors through some admissible monomorphism $\widetilde{f} : \widetilde{\mathcal{A}} \longrightarrow \mathcal{B}$ with $\mathrm{rk}_{\mathscr{C}}(\mathcal{A}) = \mathrm{rk}_{\mathscr{C}}(\widetilde{\mathcal{A}})$ and $\deg_{\mathscr{C}}(\mathcal{A}) \leq \deg_{\mathscr{C}}(\widetilde{\mathcal{A}})$.

Such a category is called a *slope category*.

We will see that the category of vector bundles on the Fargues-Fontaine curve is closely related to two other slope categories, namely the category of isocrystals and the category of filtered isocrystals. This fact will be crucial for studying the essential image of the crystalline functor in §4.2.

3.4. Semistable bundles and unramified covers

In this subsection, we construct semistable vector bundles on the Fargues-Fontaine curve by studying its unramified covers.

Definition 3.4.1. Let h be a positive integer.

(1) We denote by E_h the degree r unramified extension of $\mathbb{Q}_p$, and define the *degree h unramified cover* of X to be the natural map
$$\pi_h : X \times_{\operatorname{Spec}(\mathbb{Q}_p)} \operatorname{Spec}(E_h) \longrightarrow X.$$

(2) We write $X_h := X \times_{\operatorname{Spec}(\mathbb{Q}_p)} \operatorname{Spec}(E_h)$ and $P_h := \bigoplus_{n \geq 0} B^{\varphi^h = p^n}$.

Lemma 3.4.2. *Let r and n be integers with $r > 0$. Given a positive integer h and a nonzero homogeneous element $f \in P$, we have a canonical isomorphism*
$$B[1/f]^{\varphi^r = p^n} \otimes_{\mathbb{Q}_p} E_h \cong B[1/f]^{\varphi^{rh} = p^{nh}}.$$

Proof. The group $\operatorname{Gal}(E_h/\mathbb{Q}_p)$ is cyclic of order h, and admits a canonical generator γ which lifts the p-th power map on $\mathbb{F}_{p^h}$. Moreover, for every $n \in \mathbb{Z}$ there exists an action of $\operatorname{Gal}(E_h/\mathbb{Q}_p)$ on $B[1/f]^{\varphi^{rh} = p^{nh}}$ such that γ acts via $p^{-n}\varphi^r$. We thus find
$$B[1/f]^{\varphi^r = p^n} = \left(B[1/f]^{\varphi^{rh} = p^{nh}}\right)^{\operatorname{Gal}(E_h/\mathbb{Q}_p)},$$
and consequently deduce the desired isomorphism by the Galois descent for vector spaces. □

Proposition 3.4.3. *For every positive integer h, we have a canonical isomorphism*
$$X_h \cong \operatorname{Proj}(P_h).$$

Proof. By Lemma 3.4.2 we have $B^{\varphi = p^n} \otimes_{\mathbb{Q}_p} E_h \cong B^{\varphi^h = p^{nh}}$ for every $n \in \mathbb{Z}$, and consequently obtain a natural isomorphism
$$X_h \cong \operatorname{Proj}\left(P \otimes_{\mathbb{Q}_p} E_h\right) \cong \operatorname{Proj}\left(\bigoplus_{n \geq 0} B^{\varphi^h = p^{nh}}\right) \cong \operatorname{Proj}\left(\bigoplus_{n \geq 0} B^{\varphi^h = p^n}\right)$$
as desired. □

We invoke the following generalization of Corollary 3.1.10 without proof.

Proposition 3.4.4. *Let h and n be positive integers. Every nonzero element $f \in B^{\varphi^h = p^n}$ admits a factorization*
$$f = f_1 \cdots f_n \qquad \text{with } f_i \in B^{\varphi^h = p}$$
where the factors are uniquely determined up to $E_h^\times$-multiple.

Remark. Let us briefly sketch the proof of Proposition 3.4.4. The theory of Lubin-Tate formal groups yields a unique 1-dimensional p-divisible formal group law μ_{LT} over $\mathcal{O}_{E_h}$ with $[p]_{\mu_{\mathrm{LT}}}(t) = pt + t^{p^h}$. Denote by G_{LT} the associated p-divisible group over $\mathcal{O}_{E_h}$. By means of the logarithm for G_{LT}, we can construct a group homomorphism
$$\log_h : G_{\mathrm{LT}}(\mathcal{O}_F) := \varprojlim_i G_{\mathrm{LT}}(\mathcal{O}_F/\mathfrak{m}_F^i\mathcal{O}_F) \longrightarrow B^{\varphi^h = p}.$$
It is then not hard to extend the results from §2.3, §2.4, and §3.1 with $\log_h$, $G_{\mathrm{LT}}(\mathcal{O}_F)$, φ^h, ϕ^h, P_h, and X_h respectively taking the roles of log, $1 + \mathfrak{m}_F^*$, φ, ϕ, P, and X. We refer the readers to [**Lur**, Lecture 22-26] for details.

Definition 3.4.5. Let d and h be integers with $h > 0$. We define the *d-th twist* of $\mathcal{O}_{X_h}$ to be the quasicoherent $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d)$ associated to $P_h(d) := \bigoplus_{n\in\mathbb{Z}} B^{\varphi^h = p^{d+n}}$, where we identify $X_h \cong \text{Proj}\,(P_h)$ as in Proposition 3.4.3.

Lemma 3.4.6. *Let h be a positive integer. For every $d \in \mathbb{Z}$, the $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d)$ is a line bundle on X_h with a canonical isomorphism $\mathcal{O}_h(d) \cong \mathcal{O}_h(1)^{\otimes d}$.*

Proof. The assertion follows from Proposition 3.4.4 by a general fact as stated in [**Sta**, Tag 01MT]. □

Definition 3.4.7. Let h be a positive integer.

(1) For every positive integer r, we define the *degree r unramified cover* of X_h to be the natural map
$$\pi_{rh,h} : X_{rh} \cong X_h \times_{\text{Spec}\,(E_h)} \text{Spec}\,(E_{rh}) \longrightarrow X_h.$$
(2) For every pair of integers (d, r) with $r > 0$, we write $\mathcal{O}_h(d, r) := (\pi_{rh,h})_* \mathcal{O}_{rh}(d)$.
(3) For every nonzero homogeneous $f \in P$, we denote by $D_h(f)$ the preimage of the open subscheme $D(f) := \text{Spec}\,(B[1/f]^{\varphi=1}) \subseteq X$ under π_h.

Lemma 3.4.8. *Let h be a positive integer.*

(1) The scheme X_h is covered by open subschemes of the form $D_h(f)$ for some nonzero homogeneous element $f \in P$.
(2) Given two nonzero homogeneous f and g in P, we have $D_h(f) \cap D_h(g) = D_h(fg)$.

Proof. Both statements evidently hold for $h = 1$ as we have $X_1 = X = \text{Proj}\,(P)$. The assertion for the general case then follows by the surjectivity of π_h. □

Proposition 3.4.9. *Let $d, h,$ and r be integers with $h, r > 0$.*

(1) The $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d, r)$ is a vector bundle on X_h of rank r.
(2) Given a nonzero homogeneous $f \in P$, there exists a canonical identification
$$\mathcal{O}_h(d, r)\,(D_h(f)) \cong B[1/f]^{\varphi^{hr} = p^d}.$$

Proof. The first statement follows from Lemma 3.4.6 since the morphism $\pi_{rh,h}$ is finite of degree r. The second statement is obvious by construction. □

Proposition 3.4.10. *Let d and r be integers with $r > 0$. Given arbitrary positive integers h and n, there exists a natural identification*
$$(\pi_{hn,h})^* \mathcal{O}_h(d, r) \cong \mathcal{O}_{hn}(dn, r).$$

Proof. Let $f \in P$ be an arbitrary nonzero homogeneous element. Since $D_{hn}(f)$ is the inverse image of $D_h(f)$ under $\pi_{hn,h}$, we use Lemma 3.4.2 and Proposition 3.4.9 to find
$$\begin{aligned}(\pi_{hn,h})^* \mathcal{O}_h(d, r)\,(D_{hn}(f)) &\cong \mathcal{O}_h(d, r)\,(D_h(f)) \otimes_{B[1/f]^{\varphi^h=1}} B[1/f]^{\varphi^{hn}=1} \\ &\cong B[1/f]^{\varphi^{hr}=p^d} \otimes_{B[1/f]^{\varphi^h=1}} \left(B[1/f]^{\varphi^h=1} \otimes_{\mathbb{Q}_p} E_n\right) \\ &\cong B[1/f]^{\varphi^{hr}=p^d} \otimes_{\mathbb{Q}_p} E_n \\ &\cong B[1/f]^{\varphi^{hnr}=p^{dn}} \\ &\cong \mathcal{O}_{hn}(dn, r)\,(D_{hn}(f))\,.\end{aligned}$$
The desired assertion now follows by Lemma 3.4.8. □

Proposition 3.4.11. *Let d and r be integers with $r > 0$. Given arbitrary positive integers h and n, we have a natural isomorphism*

$$\mathcal{O}_h(dn, rn) \cong \mathcal{O}_h(d, r)^{\oplus n}.$$

Proof. By Proposition 3.4.10 we obtain a natural isomorphism

$$\mathcal{O}_h(dn, rn) = (\pi_{hr,h})_*(\pi_{hnr,hr})_*\mathcal{O}_{hnr}(dn) \cong (\pi_{hr,h})_*(\pi_{hnr,hr})_*(\pi_{hnr,hr})^*\mathcal{O}_{hr}(d).$$

Then we use the projection formula to find

$$(\pi_{hnr,hr})_*(\pi_{hnr,hr})^*\mathcal{O}_{hr}(d) \cong (\pi_{hnr,hr})_*\mathcal{O}_{X_{hnr}} \otimes_{\mathcal{O}_{X_{hr}}} \mathcal{O}_{hr}(d) \cong \mathcal{O}_{X_{hr}}^{\oplus n} \otimes_{\mathcal{O}_{X_{hr}}} \mathcal{O}_{hr}(d) \cong \mathcal{O}_{hr}(d)^{\oplus n},$$

and consequently deduce the desired assertion. □

Proposition 3.4.12. *Let h be a positive integer. We have a canonical isomorphism*

$$\mathcal{O}_h(d_1, r_1) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) \cong \mathcal{O}_h(d_1r_2 + d_2r_1, r_1r_2)$$

for all integers d_1, d_2, r_1, r_2 with $r_1, r_2 > 0$.

Proof. Let g and l respectively denote the greatest common divisor and the least common multiple of r_1 and r_2. Since r_1/g and r_2/g are relatively prime integers, the fields E_{r_1h} and E_{r_2h} are linearly disjoint finite extensions of E_{gh} with $E_{r_1h}E_{r_2h} = E_{lh}$. Hence we have an identification $E_{lh} \cong E_{r_1h} \otimes_{E_{gh}} E_{r_2h}$, which gives rise to a cartesian diagram

$$\begin{array}{ccc} X_{lh} & \xrightarrow{\pi_{lh,r_2h}} & X_{r_2h} \\ {\scriptstyle \pi_{lh,r_1h}}\downarrow & & \downarrow{\scriptstyle \pi_{r_2h,gh}} \\ X_{r_1h} & \xrightarrow{\pi_{r_1h,gh}} & X_{gh} \end{array}$$

where all arrows are finite étale. Let us now write $r_1' := r_1/g$ and $r_2' := r_2/g$. Then we find

$$\begin{aligned} \mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2, r_2') &= (\pi_{r_1h,gh})_*(\mathcal{O}_{r_1h}(d_1)) \otimes_{\mathcal{O}_{X_{gh}}} (\pi_{r_2h,gh})_*(\mathcal{O}_{r_2h}(d_2)) \\ &\cong (\pi_{lh,gh})_* \Big((\pi_{lh,r_1h})^*\mathcal{O}_{r_1h}(d_1) \otimes_{\mathcal{O}_{X_{lh}}} (\pi_{lh,r_2h})^*\mathcal{O}_{r_2h}(d_2)\Big) \\ &\cong (\pi_{lh,gh})_* \Big(\mathcal{O}_{lh}(d_1r_1') \otimes_{\mathcal{O}_{X_{lh}}} \mathcal{O}_{lh}(d_2r_2')\Big) \\ &\cong (\pi_{lh,gh})_*\mathcal{O}_{lh}(d_1r_1' + d_2r_2') \\ &= \mathcal{O}_{gh}(d_1r_1' + d_2r_2', r_1'r_2') \end{aligned}$$

where the isomorphisms respectively follow from the Künneth formula, Proposition 3.4.10, and Lemma 3.4.6. We thus use the projection formula, Proposition 3.4.10, and Proposition 3.4.11 to obtain an identification

$$\begin{aligned} \mathcal{O}_h(d_1, r_1) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) &= (\pi_{gh,h})_*\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) \\ &\cong (\pi_{gh,h})_* \Big(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} (\pi_{gh,h})^*\mathcal{O}_h(d_2, r_2)\Big) \\ &\cong (\pi_{gh,h})_* \Big(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2g, r_2)\Big) \\ &\cong (\pi_{gh,h})_* \Big(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2, r_2')^{\oplus g}\Big) \\ &\cong (\pi_{gh,h})_*\mathcal{O}_{gh}(d_1r_1' + d_2r_2', r_1'r_2')^{\oplus g} \\ &= \mathcal{O}_h(d_1r_1' + d_2r_2', gr_1r_2)^{\oplus g} \\ &\cong \mathcal{O}_h(d_1r_1 + d_2r_2, r_1r_2), \end{aligned}$$

thereby completing the proof. □

Proposition 3.4.13. *Let d and r be ingeters with $r > 0$. For every positive integer h, there exists a canonical isomorphism*

$$\mathcal{O}_h(d,r)^\vee \cong \mathcal{O}_h(-d,r).$$

Proof. Proposition 3.4.11 and Proposition 3.4.12 together yield a natural map

$$\mathcal{O}_h(d,r) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(-d,r) \cong \mathcal{O}_{X_h}^{\oplus r^2} \cong \mathcal{O}_{X_h}^{\oplus r} \otimes_{\mathcal{O}_{X_h}} \left(\mathcal{O}_{X_h}^{\oplus r}\right)^\vee \longrightarrow \mathcal{O}_{X_h}$$

where the last arrow is given by the trace map. It is straightforward to verify that this map is a perfect pairing, which in turn yields the desired isomorphism. □

Proposition 3.4.14. *Let d and r be integers with $r > 0$.*

(1) The vector bundle $\mathcal{O}(d,r) := \mathcal{O}_1(d,r)$ on X is semistable of rank r and degree d.
(2) If d and r are relatively prime, then the bundle $\mathcal{O}(d,r)$ is stable.

Proof. Proposition 3.4.11 and Proposition 3.4.12 together yield a natural isomorphism

$$\mathcal{O}(d,r)^{\otimes r} \cong \mathcal{O}(dr^r, r^r) \cong \mathcal{O}(d)^{\oplus r^r}. \tag{3.18}$$

Moreover, we find $\deg\left(\mathcal{O}(d)^{\oplus r^r}\right) = dr^r$ by Proposition 3.3.4. Therefore it follows by Proposition 3.3.6 and Proposition 3.4.9 that $\mathcal{O}(d,r)$ is of rank r and degree d. Furthermore, since $\mathcal{O}(d)$ is stable as noted in Example 3.3.14, we find by Proposition 3.3.18 that $\mathcal{O}(d)^{\oplus r^r}$ is semistable, and consequently deduce by (3.18) and Proposition 3.3.17 that $\mathcal{O}(d,r)$ is semistable as well.

Let us now assume that d and r are relatively prime. Take an arbitrary nonzero proper subbundle $\mathcal{V}$ of $\mathcal{O}(d,r)$. We have $\mu(\mathcal{V}) \neq d/r$ as $\text{rk}(\mathcal{V})$ is less than $\text{rk}(\mathcal{O}(d,r)) = r$. Hence we find $\mu(\mathcal{V}) < \lambda$ by the semistability of $\mathcal{O}(d,r)$, thereby deducing that $\mathcal{O}(d,r)$ is stable. □

Remark. Proposition 3.4.14 readily extends to $\mathcal{O}_h(d,r)$ and X_h for every positive integer h, as it turns out that X_h is a complete algebraic curve. In fact, extending the remark after Proposition 3.4.4, it is not hard to show that all results from §3.2 remain valid with φ^h, P_h, X_h, and $\mathcal{O}_h(d)$ respectively in place of φ, P, X, and $\mathcal{O}(d)$; in particular, X_h is a Dedekind scheme whose Picard group is isomorphic to $\mathbb{Z}$.

Definition 3.4.15. Let $\lambda = d/r$ be a rational number, written in a reduced form with $r > 0$. We refer to $\mathcal{O}(\lambda) := \mathcal{O}_1(d,r)$ as the *canonical stable bundle* on X of slope λ.

Proposition 3.4.16. *Let λ be a rational number.*

(1) There exists a canonical isomorphism $\mathcal{O}(\lambda)^\vee \cong \mathcal{O}(-\lambda)$.
(2) Given a rational number λ', we have a natural isomorphism

$$\mathcal{O}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda') \cong \mathcal{O}(\lambda + \lambda')^{\oplus n}$$

for some positive integer n.

Proof. The first statement is a special case of Proposition 3.4.13. The second statement follows from Proposition 3.4.11 and Proposition 3.4.12. □

Remark. By the remark after Proposition 3.4.14, for every positive integer h we can define the canonical stable bundle $\mathcal{O}_h(\lambda)$ of slope λ on X_h and extend Proposition 3.4.16 to $\mathcal{O}_h(\lambda)$.

3.5. Classification of the vector bundles

In this subsection, we describe a complete classification of vector bundles on the Fargues-Fontaine curve. We invoke the following technical result without proof.

Proposition 3.5.1. *Let λ be a rational number.*

(1) A vector bundle on X is semistable of slope λ if and only if is isomorphic to $\mathcal{O}(\lambda)^{\oplus n}$ for some $n \geq 1$.

(2) If we have $\lambda \geq 0$, the cohomology group $H^1(X, \mathcal{O}(\lambda))$ vanishes.

Remark. The second statement is relatively easy to prove. Let us write $\lambda = d/r$ where d and r are relatively prime integers with $r > 0$. As remarked after Proposition 3.4.14, Theorem 3.2.9 is valid with $\mathcal{O}_r(d)$ and X_r respectively in place of $\mathcal{O}(d)$ and X. Hence for $\lambda \geq 0$ we find

$$H^1(X, \mathcal{O}(\lambda)) = H^1(X, (\pi_r)_*\mathcal{O}_r(d)) \cong H^1(X_r, \mathcal{O}_r(d)) = 0.$$

On the other hand, the first statement is one of the most technical results from the original work of Fargues and Fontaine [**FF18**]. Here we can only sketch some key ideas for the proof. We refer the curious readers to [**FF14**, §6] for a good exposition of the proof.

The if part of the first statement is immediate by Proposition 3.4.14. In order to prove the converse, it is essential to simultaneously consider all unramified covers of X; more precisely, we assert that every semistable vector bundle $\mathcal{V}$ on X_h of slope λ is isomorphic to $\mathcal{O}_h(\lambda)^{\oplus n}$ for some $n \geq 1$, where we set $\mathcal{O}_h(\lambda) := \mathcal{O}_h(d, r)$. The proof of this statement is given by a series of dévissage arguments as follows:

(a) We may replace $\mathcal{V}$ with $(\pi_{rh,r})^*\mathcal{V}$ to assume that λ is an integer; this reduction is based on the identification $(\pi_{rh,r})_*(\pi_{rh,r})^*\mathcal{O}_h(\lambda) \cong \mathcal{O}_h(d)^{\oplus r}$ given by Proposition 3.4.10 and the fact that $(\pi_{rh,r})^*\mathcal{V}$ is semistable of slope d as seen by an elementary Galois descent argument based on Theorem 3.3.22.

(b) We may replace $\mathcal{V}$ by $\mathcal{V}(-\lambda) := \mathcal{V} \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(-\lambda)$ to further assume $\lambda = 0$; this reduction is based on the identification $\mathcal{O}_h(\lambda) \cong \mathcal{O}_h \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h$ and the fact that $\mathcal{V}(-\lambda)$ is semistable of slope 0 as easily seen by Proposition 3.3.6.

(c) With $\lambda = 0$, it suffices to prove that $H^0(X_h, \mathcal{V})$ does not vanish; indeed, any nonzero global section of $\mathcal{V}$ gives rise to an exact sequence of vector bundles on X_h

$$0 \longrightarrow \mathcal{O}_{X_h} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0$$

where $\mathcal{W}$ is semistable of slope 0 by Proposition 3.3.18, thereby allowing us to proceed by induction on $\mathrm{rk}(\mathcal{V})$ with the identification $\mathrm{Ext}^1_{\mathcal{O}_{X_h}}(\mathcal{O}_h, \mathcal{O}_h) \cong H^1(X_h, \mathcal{O}_{X_h}) = 0$.

(d) The proof further reduces to the case where $\mathcal{V}$ fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}_h(-1/n) \longrightarrow \mathcal{V} \longrightarrow \mathcal{O}_h(1) \longrightarrow 0$$

with $n = \mathrm{rk}(\mathcal{V}) - 1$; this reduction involves a generalization of Grothendieck's argument for the classification of vector bundles on the projective line.

(e) The exact sequence above turns out to naturally arise from p-divisible groups, as we will remark after Example 3.5.4; as a consequence the assertion eventually follows from some results about period morphisms on the Lubin-Tate spaces due to Drinfeld [**Dri76**], Gross-Hopkins [**GH94**], and Laffaille [**Laf85**].

Corollary 3.5.2. *The tensor product of two semistable vector bundles on X is semistable.*

Proof. This is an immediate consequence of Proposition 3.4.16 and Proposition 3.5.1. □

Theorem 3.5.3 (Fargues-Fontaine [**FF18**])**.** *Every vector bundle $\mathcal{V}$ on X admits a unique Harder-Narasimhan filtration*

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V},$$

which (noncanonically) splits into a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i}$$

where we set $\lambda_i := \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$ for each $i = 1, \cdots, n$.

Proof. Existence and uniqueness of the Harder-Narasimhan filtration is an immediate consequence of Theorem 3.3.22. Hence it remains to prove that the Harder-Narasimhan filtration splits. Let us proceed by induction on n. If we have $n = 0$, then the assertion is trivial. We henceforth assume $n > 0$. By construction each successive quotient $\mathcal{V}_i/\mathcal{V}_{i-1}$ is semistable of slope λ_i. Hence Proposition 3.5.1 yields an isomorphism

$$\mathcal{V}_i/\mathcal{V}_{i-1} \simeq \mathcal{O}(\lambda_i)^{\oplus m_i} \qquad \text{for each } i = 1, \cdots, n \tag{3.19}$$

where m_i is a positive integer. Moreover, by the induction hypothesis, the filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{n-1}$$

splits into a direct sum decomposition

$$\mathcal{V}_{l-1} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(\lambda_i)^{\oplus m_i}. \tag{3.20}$$

Hence it suffices to establish the identity

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{V}/\mathcal{V}_{n-1}, \mathcal{V}_{n-1}) = 0. \tag{3.21}$$

For each $i = 1, \cdots, n$, Proposition 3.4.16 yields an identification

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(\lambda_n), \mathcal{O}(\lambda_i)) \cong H^1(X, \mathcal{O}(\lambda_i) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda_n)^\vee) \cong H^1(X, \mathcal{O}(\lambda_i - \lambda_n)^{\oplus n_i})$$

where n_i is a positive integer. Since we have $\lambda_i \geq \lambda_n$ for each $i = 1, \cdots, n$, we find

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(\lambda_n), \mathcal{O}(\lambda_i)) = 0 \qquad \text{for each } i = 1, \cdots, n$$

by Proposition 3.5.1. Therefore we deduce the identity (3.21) by the decompositions (3.19) and (3.20), thereby completing the proof. □

Remark. Theorem 3.5.3 is an analogue of the fact that every vector bundle $\mathcal{W}$ on the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ admits a direct sum decomposition

$$\mathcal{W} \simeq \bigoplus_{j=1}^{l} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d_j)^{\oplus k_j} \qquad \text{with } d_j \in \mathbb{Z}.$$

The only essential difference is that semistable vector bundles on X may have rational slopes, whereas semistable vector bundles on $\mathbb{P}^1_{\mathbb{C}}$ have integer slopes. This difference comes from the fact that we have $H^1(X, \mathcal{O}(-1)) \neq 0$ and $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)) = 0$ as remarked after Theorem 3.2.9.

It is worthwhile to mention that an equivalent result of Theorem 3.5.3 was first obtained by Kedlaya [**Ked05**]. In fact, Kedlaya's result can be reformulated as a classification of vector bundles on the adic Fargues-Fontaine curve, which recovers Theorem 3.5.3 by Theorem 1.3.24.

Example 3.5.4. Let us write $W(\overline{\mathbb{F}}_p)$ for the ring of Witt vectors over $\overline{\mathbb{F}}_p$, and K_0 for the fraction field of $W(\overline{\mathbb{F}}_p)$. Let N be an isocrystal over K_0 which admits a decomposition

$$N \simeq \bigoplus_{i=1}^{n} N(\lambda_i)^{\oplus m_i} \qquad \text{with } \lambda_i \in \mathbb{Q}. \tag{3.22}$$

We assert that N naturally gives rise to a vector bundle $\mathcal{E}(N)$ on X with an isomorphism

$$\mathcal{E}(N) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i}. \tag{3.23}$$

We may regard K_0 as a subring of B under the identification

$$K_0 = W(\overline{\mathbb{F}}_p)[1/p] \cong \left\{ \sum [c_n]p^n \in A_{\mathrm{inf}}[1/p] : c_n \in \overline{\mathbb{F}}_p \right\}.$$

Then by construction φ restricts to the Frobenius automorphism of K_0, and thus acts on N and $N^\vee$ via the Frobenius automorphisms φ_N and $\varphi_{N^\vee}$. Hence we get a graded P-module

$$P(N) := \bigoplus_{n \geq 0} (N^\vee \otimes_{K_0} B)^{\varphi = p^n}.$$

Let us set $\mathcal{E}(N)$ to be the associated quasicoherent sheaf on X, and take an arbitrary nonzero homogeneous element $f \in P$. In addition, for each $i = 1, \cdots, n$, we write $\lambda_i := d_i/r_i$ where d_i and r_i are relatively prime integers with $r_i > 0$. By construction we have

$$\begin{aligned} \mathcal{E}(N)(D(f)) &\cong \left(N^\vee \otimes_{K_0} B[1/f]\right)^{\varphi=1} = (\mathrm{Hom}_{K_0}(N, K_0) \otimes_{K_0} B[1/f])^{\varphi=1} \\ &\cong \mathrm{Hom}_{K_0}(N, B[1/f])^{\varphi=1}. \end{aligned} \tag{3.24}$$

Moreover, since each $N(\lambda_i)$ admits a basis $(\varphi^j(n))$ for some $n \in N(\lambda_i)$ with $\varphi^{r_i}(n) = p^{d_i}n$, there exists an identification

$$\mathrm{Hom}_{K_0}(N(\lambda_i), B[1/f])^{\varphi=1} \cong B[1/f]^{\varphi^{r_i}=p^{d_i}} \cong \mathcal{O}(\lambda_i)(D(f)) \tag{3.25}$$

where the last isomorphism follows from Proposition 3.4.9. As $f \in P$ is arbitrarily chosen, we obtain the isomorphism (3.23) by (3.22), (3.24) and (3.25).

Remark. As noted in Chapter II, Theorem 2.3.15, every isocrystal over K_0 admits a direct sum decomposition as in (3.22). Hence by Theorem 3.5.3 and Example 3.5.4 we obtain an essentially surjective functor

$$\mathcal{E} : \varphi\text{−}\mathrm{Mod}_{K_0} \longrightarrow \mathrm{Bun}_X$$

where $\varphi\text{−}\mathrm{Mod}_{K_0}$ and Bun_X respectively denote the category of isocrystals over K_0 and the category of vector bundles on X. Furthremore, if we have $0 \leq \lambda_i \leq 1$ for each $i = 1, \cdots, n$, then Proposition 2.3.18 from Chapter II yields a p-divisible group G over $\overline{\mathbb{F}}_p$ with

$$\mathcal{E}(\mathbb{D}(G)[1/p]) \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i}.$$

However, the functor $\mathcal{E}$ is not an equivalence of categories; indeed, for arbitrary rational numbers κ and λ with $\kappa < \lambda$, we have

$$\mathrm{Hom}_{\varphi\text{−}\mathrm{Mod}_{K_0}}(N(\kappa), N(\lambda)) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}(N(\kappa)), \mathcal{E}(N(\lambda))) \neq 0.$$

4. Applications to p-adic representations

In this section, we prove some fundamental results about p-adic representations and period rings by exploiting our accumulated knowledge of the Fargues-Fontaine curve. The primary references for this section are Fargues and Fontaine's survey paper [**FF12**] and Morrow's notes [**Mor**].

4.1. Geometrization of p-adic period rings

Throughout this section, we let K be a p-adic field with the absolute Galois group Γ_K, the inertia group I_K and the residue field k. We also write $W(k)$ for the ring of Witt vectors over k, and K_0 for its fraction field.

Proposition 4.1.1. *The tilt of $\mathbb{C}_K$ is algebraically closed.*

Proof. Let $f(x)$ be an arbitrary monic polynomial of degree $d > 0$ over $\mathbb{C}_K^\flat$. We wish to show that $f(x)$ has a root in $\mathbb{C}_K^\flat$. Take an element m in the maximal ideal of $\mathcal{O}_{\mathbb{C}_K^\flat}$. We may replace $f(x)$ by $m^{nd}f(x/m^n)$ for some sufficiently large n to assume that $f(x)$ is a polynomial over $\mathcal{O}_{\mathbb{C}_K^\flat}$. Moreover, we may assume $d > 1$ since otherwise the assertion would be obvious. Let us now write

$$f(x) = x^d + c_1x^{d-1} + \cdots + c_d \qquad \text{with } c_i \in \mathcal{O}_{\mathbb{C}_K^\flat}.$$

Proposition 2.1.7 and Proposition 2.1.8 from Chapter III together yield an identification

$$\mathcal{O}_{\mathbb{C}_K^\flat} \cong \varprojlim_{c \mapsto c^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}. \tag{4.1}$$

Write $(c_{i,n})$ for the image of each c_i under this isomorphism, and choose a lift $\widetilde{c_{i,n}} \in \mathcal{O}_{\mathbb{C}_K}$ of each $c_{i,n}$. In addition, for each $n \geq 0$ we set

$$f_n(x) := x^d + c_{1,n}x^{d-1} + \cdots + c_{d,n} \qquad \text{and} \qquad \widetilde{f_n}(x) := x^d + \widetilde{c_{1,n}}x^{d-1} + \cdots + \widetilde{c_{d,n}}.$$

Then for each $n \geq 1$ we have

$$f_{n-1}(x^p) = x^{dp} + c_{1,n}^p x^{(d-1)p} + \cdots + c_{d,n}^p = \left(x^d + c_{1,n}x^{d-1} + \cdots + c_{d,n}\right)^p = f_n(x)^p. \tag{4.2}$$

Moreover, since $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.5, each $\widetilde{f_n}(x)$ admits a factorization

$$\widetilde{f_n}(x) = (x - \alpha_{n,1}) \cdots (x - \alpha_{n,d}) \qquad \text{with } \alpha_{n,j} \in \mathcal{O}_{\mathbb{C}_K}.$$

Let us denote by $\overline{\alpha_{n,j}}$ the image of each $\alpha_{n,j}$ under the natural surjection $\mathcal{O}_{\mathbb{C}_K} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. For each $\overline{\alpha_{n,j}}$ with $n \geq 1$ we obtain $f_{n-1}(\overline{\alpha_{n,j}}^p) = f_n(\overline{\alpha_{n,j}})^p = 0$ by (4.2), and in turn find

$$\widetilde{f_{n-1}}(\alpha_{n,j}^p) = (\alpha_{n,j}^p - \alpha_{n-1,1}) \cdots (\alpha_{n,j}^p - \alpha_{n-1,d}) \in p\mathcal{O}_{\mathbb{C}_K}.$$

Hence for each $\alpha_{n,j}$ with $n \geq 1$ we have $\alpha_{n,j}^p - \alpha_{n-1,l} \in p^{1/d}\mathcal{O}_{\mathbb{C}_K}$ for some l, and consequently obtain $\overline{\alpha_{n,j}}^{p^d} = \overline{\alpha_{n-1,l}}^{p^{d-1}}$ by Lemma 2.1.6 from Chapter III. It follows that there exists a sequence of integers (j_n) with $\overline{\alpha_{n,j_n}}^{p^d} = \overline{\alpha_{n-1,j_{n-1}}}^{p^{d-1}}$ for all $n \geq 1$. Let us now set $\overline{\alpha} := \left(\overline{\alpha_{n+d-1,j_{n+d-1}}}^{p^{d-1}}\right)$. Then under the identification (4.1) we find

$$f(\overline{\alpha}) = \left(f_n\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}^{p^{d-1}}\right)\right) = \left(f_{n+d-1}\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}\right)\right) = 0$$

where the second identity follows by (4.2). $\square$

Remark. Our proof above readily extends to show that the tilt of an algebraically closed perfectoid field is algebraically closed.

For the rest of this section, we take $F = \mathbb{C}_K^\flat$ and regard $\mathbb{C}_K$ as an untilt of F. We also fix an element $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$ and set $\xi := [p^\flat] - p \in A_{\text{inf}}$. In addition, we choose a valuation ν_F on F with $\nu_F(p^\flat) = 1$.

Proposition 4.1.2. *Let ε be an element in $\mathcal{O}_F$ with $\varepsilon^\sharp = 1$ and $(\varepsilon^{1/p})^\sharp \neq 1$.*

(1) We have $\varepsilon \in 1 + \mathfrak{m}_F^*$.

(2) The element $t := \log(\varepsilon) \in B^{\varphi=p}$ is a prime in P, and gives rise to a closed point ∞ on X with the following properties:

(i) The residue field at ∞ is naturally isomorphic to $\mathbb{C}_K$.

(ii) The completed local ring at ∞ is naturally isomorphic to B_{dR}^+.

Proof. The first statement is an immediate consequence of Lemma 2.2.17 from Chapter III (or the proof of Proposition 2.3.3). We then observe by Proposition 2.3.3 that $t = \log(\varepsilon)$ vanishes at an element $y_\infty \in Y$ represented by $\mathbb{C}_K$, and consequently deduce the second statement from Proposition 2.4.7 and Theorem 2.4.8. □

Proposition 4.1.3. *There exists a natural isomorphism*

$$B_{\text{dR}}^+ \cong \varprojlim_j B/\ker(\widehat{\theta_{\mathbb{C}_K}})^j \tag{4.3}$$

which induces a topology on B_{dR}^+ with the following properties:

(i) The subring A_{inf} of B_{dR}^+ is closed.

(ii) The map $\theta_{\mathbb{C}_K}[1/p] : A_{\text{inf}}[1/p] \twoheadrightarrow \mathbb{C}_K$ induced by $\theta_{\mathbb{C}_K}$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.

(iii) The logarithm on $1 + \mathfrak{m}_F$ induces a continuous map $\log : \mathbb{Z}_p(1) \longrightarrow B_{\text{dR}}^+$ under the natural identification $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K}) = \{\, \varepsilon \in \mathcal{O}_F : \varepsilon^\sharp = 1 \,\}$.

(iv) The multiplication by any uniformizer yields a closed embedding on B_{dR}^+.

(v) The ring B_{dR}^+ is complete.

Proof. The natural isomorphism (4.3) is given by Proposition 2.2.7. Let us equip B_{dR}^+ with the inverse limit topology via (4.3). The property (ii) follows from Proposition 1.2.16 and the fact that $\theta_{\mathbb{C}_K}[1/p]$ extends to $\widehat{\theta_{\mathbb{C}_K}}$. The property (iii) is evident by Proposition 3.1.8.

Let us now establish the property (i). Recall that we may regard $A_{\text{inf}}[1/p]$ as a subring of B_{dR}^+ in light of Corollary 2.2.11 from Chapter III. Proposition 3.1.4 implies that A_{inf} is complete with respect to all Gauss norms. Moreover, by Example 2.1.6 we have $|\xi|_\rho < 1$ for all $\rho \in (0, 1)$, and consequently find that every ξ-adically Cauchy sequence in A_{inf} is also Cauchy with respect to all Gauss norms. We then deduce the assertion by the fact that ξ generates $\ker(\widehat{\theta_{\mathbb{C}_K}})$ as noted in Corollary 2.2.4.

It remains to verify the properties (iv) and (iv). We find by Proposition 1.2.16 that $\ker(\widehat{\theta_{\mathbb{C}_K}}) = \xi B$ is closed in B, and in turn deduce that $\ker(\widehat{\theta_{\mathbb{C}_K}})^j = \xi^j B$ is closed in B for each $j \geq 1$. Hence the property (iv) follows by the fact that every uniformizer of B_{dR}^+ is a unit multiple of ξ as noted in Proposition 2.2.7. In addition, we find by the completeness of B that $B/\ker(\widehat{\theta_{\mathbb{C}_K}})^j$ is complete for each $j \geq 1$, and consequently obtain the property (iv). □

Remark. Proposition 4.1.3 proves Proposition 2.2.16 from Chapter III. Our proof does not rely on any unproved results such as Proposition 2.4.1, Proposition 3.4.4 or Proposition 3.5.1.

We henceforth fix $\varepsilon \in 1 + \mathfrak{m}_F^*$, $t \in B^{\varphi=p}$ and $\infty \in |X|$ as in Proposition 4.1.2. We also write B^+ for the closure of $A_{\text{inf}}[1/p]$ in B. In addition, for every $\rho \in (0,1)$ we denote by B^+_ρ the closure of $A_{\text{inf}}[1/p]$ in $B_{[\rho,\rho]}$.

Lemma 4.1.4. *Let V be a normed space over $\mathbb{Q}_p$, and let $\widehat{V_0}$ denote the p-adic completion of the closed unit disk V_0 in V. The completion of V with respect to its norm is naturally isomorphic to $\widehat{V_0}[1/p]$.*

Proof. Since p is topologically nilpotent in $\mathbb{Q}_p$, we have a neighborhood basis for $0 \in V$ given by the sets $p^n V_0$ for $n \geq 0$. This implies that a sequence in V_0 is Cauchy with respect to the norm on V if and only if it is p-adically Cauchy. Hence $\widehat{V_0}$ coincides with the completion of V_0 with respect to the norm on V. The assertion now follows by the fact that every Cauchy sequence in V becomes a Cauchy sequence in V_0 after a multiplication by some power of p. □

Remark. The notion of p-adic completion is not meaningful for V, as we have $p^n V = V$ for all $n \geq 0$.

Proposition 4.1.5. *Let c be an element in $\mathcal{O}_F^\times$. There exists a canonical continuous isomorphism*

$$B^+_{|c|} \cong \widehat{A_{\text{inf}}[[c]/p]}[1/p]$$

where $\widehat{A_{\text{inf}}[[c]/p]}$ denotes the p-adic completion of $A_{\text{inf}}[[c]/p]$.

Proof. By construction, the topological ring $B^+_{|c|}$ is naturally isomorphic to the completion of $A_{\text{inf}}[1/p]$ with respect to the Gauss $|c|$-norm. In light of Lemma 4.1.4, it is thus sufficient to establish the identification

$$A_{\text{inf}}[[c]/p] = \left\{ f \in A_{\text{inf}}[1/p] : |f|_{|c|} \leq 1 \right\}.$$

Since we have $|[c]/p|_{|c|} = 1$, the ring $A_{\text{inf}}[[c]/p]$ is contained in the set on the right hand side. Let us now consider an arbitrary element $f \in A_{\text{inf}}[1/p]$ with $|f|_{|c|} \leq 1$. We wish to show that f belongs to $A_{\text{inf}}[[c]/p]$. Let us write the Teichmüller expansion of f as

$$f = \sum_{n<0} [c_n]p^n + \sum_{n\geq 0} [c_n]p^n \qquad \text{with } c_n \in \mathcal{O}_F \tag{4.4}$$

where the first summation on the right hand side contains only finitely many nonzero terms. For every $n \in \mathbb{Z}$ we find $|c_n|\,|c|^n \leq |f|_{|c|} = 1$, or equivalently $|c_n| \leq |c|^{-n}$. Hence for every $n < 0$ we have $c_n = c^{-n} d_n$ for some $d_n \in \mathcal{O}_F$, and consequently obtain

$$[c_n]p^n = [d_n] \cdot ([c]/p)^{-n} \in A_{\text{inf}}[[c]/p].$$

The assertion is now evident by (4.4). □

Remark. Given two elements $c, d \in \mathcal{O}_F^\times$ with $|c| \leq |d|$, we can argue as above to obtain an identification

$$B_{[|c|,|d|]} \cong \widehat{A_{\text{inf}}[[c]/p, p/[d]]}[1/p]$$

where $\widehat{A_{\text{inf}}[[c]/p, p/[d]]}$ denotes the p-adic completion of $A_{\text{inf}}[[c]/p, p/[d]]$. This is in some sense reminiscent of our discussion in Example 1.3.13, which shows that for arbitrary positive real numbers $i, j \in \mathbb{Z}[1/p]$ the ring $B_{[|\varpi|^i, |\varpi|^j]}$ coincides with the completion of $A_{\text{inf}}[1/p, 1/[\varpi]]$ with respect to the ideal I generated by $[\varpi^i]/p$ and $p/[\varpi^j]$. We can use the above identification to show that the natural map $B \longrightarrow B^+_{\text{dR}}$ extends to a map $B_{[a,b]} \longrightarrow B^+_{\text{dR}}$ for any closed interval $[a,b] \subseteq (0,1)$.

Proposition 4.1.6. *We have natural continuous embeddings*

$$B^+_{1/p^p} \hookrightarrow B^+_{\mathrm{cris}} \hookrightarrow B^+_{1/p}.$$

PROOF. Let A^0_{cris} be the A_{inf}-subalgebra in $A_{\mathrm{inf}}[1/p]$ generated by the elements of the form $\xi^n/n!$ with $n \geq 0$. By definition we have $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$, where A_{cris} is naturally isomorphic to the p-adic completion of A^0_{cris} as noted in Chapter III, Proposition 3.1.9. Moreover, Proposition 4.1.5 yields natural identifications

$$B^+_{1/p^p} \cong \widehat{A_{\mathrm{inf}}[[(p^\flat)^p]/p]}[1/p] \qquad \text{and} \qquad B^+_{1/p} \cong \widehat{A_{\mathrm{inf}}[[p^\flat]/p]}[1/p],$$

where $\widehat{A_{\mathrm{inf}}[[(p^\flat)^p]/p]}$ and $\widehat{A_{\mathrm{inf}}[[p^\flat]/p]}$ respectively denote the p-adic completions of $A_{\mathrm{inf}}[[(p^\flat)^p]/p]$ and $A_{\mathrm{inf}}[[p^\flat]/p]$. Hence it suffices to show

$$A_{\mathrm{inf}}[[(p^\flat)^p]/p] \subseteq A^0_{\mathrm{cris}} \subseteq A_{\mathrm{inf}}[[p^\flat]/p]. \tag{4.5}$$

We obtain the first inclusion in (4.5) by observing

$$\frac{[p^\flat]^p}{p} = \frac{(\xi+p)^p}{p} = (p-1)! \cdot \frac{\xi}{p!} + \sum_{i=1}^{p} \binom{p}{i} p^{i-1}\xi^{p-i} \in A^0_{\mathrm{cris}}.$$

In addition, we find

$$\frac{\xi^n}{n!} = \frac{([p^\flat]-p)^n}{n!} = \frac{p^n}{n!}\left(\frac{[p^\flat]}{p} - 1\right)^n \in A_{\mathrm{inf}}[[p^\flat]/p] \qquad \text{for all } n \geq 0$$

as $p^n/n!$ is an element of $\mathbb{Z}_p$, and consequently deduce the second inclusion in (4.5). □

Lemma 4.1.7. *Let $[a,b]$ be a closed subinterval of $(0,1)$. There exists some $e > 0$ with*

$$|f|_b \leq |f|_a^e \qquad \textit{for every } f \in A_{\mathrm{inf}}[1/p].$$

PROOF. Let us set $l := -\log_p(b)$ and $r := -\log_p(a)$. Since $\mathcal{L}_f$ is a concave piecewise linear function as noted in Corollary 2.1.11, its graph on $(0,l]$ should be bounded above by the line which passes through the points $(l, \mathcal{L}_f(l))$ and $(r, \mathcal{L}_f(r))$. Hence we have

$$\mathcal{L}_f(s) \leq \frac{\mathcal{L}_f(r) - \mathcal{L}_f(l)}{r-l}(s-l) + \mathcal{L}_l \qquad \text{for all } s \in (0,l],$$

and consequently find

$$\lim_{s\to 0} \mathcal{L}_f(s) \leq \frac{-l(\mathcal{L}_f(r) - \mathcal{L}_f(l))}{r-l} + \mathcal{L}_l = \frac{-l\mathcal{L}_f(r) + r\mathcal{L}_f(l)}{r-l}.$$

Meanwhile, Proposition 3.1.4 yields an integer n with

$$\mathcal{L}_f(s) = -\log_p\left(|f|_{p^{-s}}\right) \geq -\log_p(p^{-ns}) = ns \qquad \text{for all } s \in (0,\infty),$$

and in turn implies $\lim\limits_{s\to 0} \mathcal{L}_f(s) \geq 0$. We thus obtain $r\mathcal{L}_f(l) \geq l\mathcal{L}_f(r)$, and consequently find

$$|f|_b = p^{-\mathcal{L}_f(r)} \leq p^{-(r/l)\mathcal{L}_f(l)} = |f|_a^{r/l}$$

as desired. □

Proposition 4.1.8. *For every closed interval $[a,b] \subseteq (0,1)$, there exists a canonical continuous embedding $B^+_a \hookrightarrow B^+_b$.*

PROOF. Lemma 4.1.7 implies that every Cauchy sequence in $A_{\mathrm{inf}}[1/p]$ with respect to the Gauss a-norm is Cauchy with respect to the Gauss b-norm. Hence the assertion is evident by construction. □

For the rest of this section, we write $\widetilde{B^+} := \varinjlim B_\rho^+$ where the transition maps are the natural injective maps given by Proposition 4.1.8, and regard each B_ρ^+ as a subring of $\widetilde{B^+}$. We also regard B_{cris}^+ as a subring of $\widetilde{B^+}$ in light of Proposition 4.1.6.

Proposition 4.1.9. *The Frobenius automorphism of $A_{\mathrm{inf}}[1/p]$ uniquely extends to an automorphism φ^+ of $\widetilde{B^+}$ with the following properties:*

(i) φ and φ^+ agree on B^+.
(ii) The Frobenius endomorphism of B_{cris} and φ^+ agree on B_{cris}^+.
(iii) φ^+ restricts to an isomorphism $B_\rho^+ \simeq B_{\rho^p}^+$ for every $\rho \in (0,1)$.

Proof. Let φ_{inf} denote the Frobenius automorphism of $A_{\mathrm{inf}}[1/p]$. Then we have

$$\varphi_{\mathrm{inf}}\left(\sum [c_n]p^n\right) = \sum [c_n^p]p^n \qquad \text{for all } c_n \in \mathcal{O}_F,$$

and consequently find

$$|\varphi_{\mathrm{inf}}(f)|_{\rho^p} = |f|_\rho^p \qquad \text{for all } f \in A_{\mathrm{inf}}[1/p] \text{ and } \rho \in (0,1).$$

It follows by Lemma 1.2.15 that φ_{inf} uniquely extends to a continuous ring isomorphism $\varphi_\rho^+ : B_\rho^+ \simeq B_{\rho^p}^+$ for each $\rho \in (0,1)$. For every closed subinterval $[a,b]$ of $(0,1)$, the restriction of φ_b^+ on B_a^+ is a continuous extension of φ_{inf}, and thus agrees with φ_a^+. Hence we obtain an isomorphism

$$\varphi^+ : \widetilde{B^+} = \varinjlim B_\rho^+ \simeq \varinjlim B_{\rho^p}^+ = \widetilde{B^+}.$$

It is evident by construction that φ^+ is an extension of φ_{inf} and each B_ρ^+ with $\rho \in (0,1)$. The uniqueness of each φ_ρ^+ implies that φ^+ is a unique extension of φ_{inf} with the property (iii). Moreover, the restriction of φ^+ on B_{cris}^+ is a continuous extension of φ_{inf}, and thus agrees with the Frobenius endomorphism on B_{cris}^+ by Lemma 3.1.10 from Chapter III.

It remains to verify the property (i) of φ^+. By construction, both φ and φ^+ extend φ_{inf}. In addition, the property (iii) implies that φ^+ restricts to an isomorphism

$$B^+ = \varprojlim B_\rho^+ \simeq \varprojlim B_{\rho^p}^+ = B^+$$

where the transition maps in each limit are the natural inclusions. Since B^+ is the closure of $A_{\mathrm{inf}}[1/p]$ in B, we deduce that this isomorphism agrees with the restriction of φ on B^+, thereby completing the proof. □

Remark. Let us give an alternative description of the ring $\widetilde{B^+}$ and its Frobenius automorphism. We define the Gauss 1-norm on $A_{\mathrm{inf}}[1/p]$ by

$$\left|\sum [c_n]p^n\right|_1 := \sup_{n\in\mathbb{Z}}(|c_n|) \qquad \text{for all } c_n \in \mathcal{O}_F.$$

By construction we have $|f|_1 = \lim\limits_{\rho\to 1} |f|_\rho$ for every $f \in A_{\mathrm{inf}}[1/p]$, and consequently find that the Gauss 1-norm is indeed a multiplicative norm. It is then straightforward to verify that $\widetilde{B^+}$ is naturally isomorphic to the completion of $A_{\mathrm{inf}}[1/p]$ with respect to the Gauss 1-norm. Hence we may obtain φ^+ as a unique continuous extension of φ_{inf} by Lemma 1.2.15.

However, we avoid using this description because working with the Gauss 1-norm is often subtle. The main issue is that the natural map $\mathcal{O}_F \longrightarrow A_{\mathrm{inf}}[1/p]$ given by the Teichmüler lifts is not continuous with respect to the Gauss 1-norm. In fact, it is not hard to show

$$\lim_{c\to 0} |[1+c]-1|_1 = 1 \neq 0.$$

Definition 4.1.10. We refer to the map φ^+ constructed in Proposition 4.1.9 as the *Frobenius automorphism* of $\widetilde{B^+}$. We often abuse notation and write φ for φ^+ and the Frobenius endomorphism of B_{cris}.

Proposition 4.1.11. *The Frobenius endomorphism of B_{cris} is injective.*

Proof. Proposition 4.1.9 implies that φ is injective on B^+_{cris}, and in turn yields the desired assertion as we have $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t]$ and $\varphi(t) = pt$ by Proposition 3.1.11 from Chapter III. ☐

Remark. Proposition 4.1.11 proves Theorem 3.1.13 from Chapter III.

Proposition 4.1.12. *We have identities*

$$B^+ = \bigcap_{n\geq 0} \varphi^n(B^+_{\mathrm{cris}}) \qquad \textit{and} \qquad B^+[1/t] = \bigcap_{n\geq 0} \varphi^n(B_{\mathrm{cris}}).$$

Proof. By Proposition 4.1.6 and Proposition 4.1.9 we have

$$B^+_{1/p^{p^{n+1}}} = \varphi^n(B^+_{1/p^p}) \subseteq \varphi^n(B^+_{\mathrm{cris}}) \subseteq \varphi^n(B^+_{1/p}) = B^+_{1/p^{p^n}} \qquad \text{for every } n \geq 0,$$

and consequently find

$$B^+ = \bigcap_{\rho\geq 0} B^+_\rho = \bigcap_{n\geq 0} B^+_{1/p^{p^n}} = \bigcap_{n\geq 0} \varphi^n(B^+_{\mathrm{cris}}).$$

The second identity then follows as we have $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t]$ and $\varphi(t) = pt$ by Proposition 3.1.11 from Chapter III. ☐

Proposition 4.1.13. *For every $n \in \mathbb{Z}$, we have*

$$B^{\varphi=p^n} = (B^+)^{\varphi=p^n} = (B^+_{\mathrm{cris}})^{\varphi=p^n}.$$

Proof. The first identity is an immediate consequence of Proposition 3.1.11. The second identity follows from Proposition 4.1.12. ☐

Corollary 4.1.14. *We have* $X = \mathrm{Proj}\left(\bigoplus_{n\geq 0} (B^+_{\mathrm{cris}})^{\varphi=p^n}\right)$.

Remark. Corollary 4.1.14 recovers the first definition of the Fargues-Fontaine curve as given in Chapter I, Definition 2.1.1.

Proposition 4.1.15. *There exists a canonical isomorphism $B_e \cong B[1/t]^{\varphi=1}$.*

Proof. Proposition 4.1.12 and Proposition 4.1.13 together yield a natural identification

$$B[1/t]^{\varphi=1} \cong B^+[1/t]^{\varphi=1} = B^{\varphi=1}_{\mathrm{cris}} = B_e$$

as desired. ☐

Corollary 4.1.16. *The ring B_e is a principal ideal domain.*

Proof. By construction, the element t induces the closed point ∞ on X. Hence we have an identification $X \setminus \{\infty\} \cong \mathrm{Spec}\,(B[1/t]^{\varphi=1})$, and consequently deduce the assertion by Theorem 2.4.8. ☐

Remark. Corollary 4.1.16 was first proved by Fontaine prior to the construction of the Fargues-Fontaine curve. Fontaine's proof was motivated by a result by Berger [**Ber08**] that B_e is a Bézout ring, and eventually inspired the first construction of the Fargues-Fontaine curve as we will soon describe in the subsequent subsection.

4.2. Essential image of the crystalline functor

In this subsection, we describe the essential image of the functor D_{cris} using vector bundles on the Fargues-Fontaine curve. Our discussion will be cursory, and will focus on explaining some key ideas for studying p-adic Galois representations via vector bundles on the Fargues-Fontaine curve. Throughout this subsection, let us write $U := X \setminus \{ \infty \}$.

Proposition 4.2.1. *Let M_e be a free B_e-module of finite rank, and let M_{dR}^+ be a B_{dR}^+-lattice in $M_{\text{dR}} := M_e \otimes_{B_e} B_{\text{dR}}$.*

(1) There exists a unique vector bundle $\mathcal{V}$ on X with

$$H^0(U, \mathcal{V}) \cong M_e \qquad \textit{and} \qquad \widehat{\mathcal{V}_\infty} \cong M_{\text{dR}}^+$$

where $\widehat{\mathcal{V}_\infty}$ denotes the completed stalk of $\mathcal{V}$ at ∞.

(2) The vector bundle $\mathcal{V}$ gives rise to a natural exact sequence

$$0 \longrightarrow H^0(X, \mathcal{V}) \longrightarrow M_e \oplus M_{\text{dR}}^+ \longrightarrow M_{\text{dR}} \longrightarrow H^1(X, \mathcal{V}) \longrightarrow 0$$

where the middle arrow maps each (x, y) to $x - y$.

Remark. The first statement is in fact a standard application of the Beauville-Laszlo theorem as stated in [**BL95**] or [**Sta**, Tag 0BP2]. The second statement then follows as a variant of the Mayer-Vietoris long exact sequence.

Example 4.2.2. By Proposition 4.1.2 and Proposition 4.1.15 we have natural identifications

$$H^0(U, \mathcal{O}_X) \cong B_e \qquad \text{and} \qquad \widehat{\mathcal{O}_{X,\infty}} \cong B_{\text{dR}}^+$$

where $\widehat{\mathcal{O}_{X,\infty}}$ denotes the completed local ring at ∞. Hence by Therem 3.2.9 and Proposition 4.2.1 we obtain a natural exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \oplus B_{\text{dR}}^+ \longrightarrow B_{\text{dR}} \longrightarrow 0,$$

which in turn yields the fundamental exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\text{dR}}/B_{\text{dR}}^+ \longrightarrow 0$$

as described in Chapter III, Theorem 3.1.14.

Remark. In fact, the Fargues-Fontaine curve was originally constructed by gluing $\text{Spec}\,(B_e)$ and $\text{Spec}\,(B_{\text{dR}}^+)$ using the fundamental exact sequence, partially motivated by Colmez's theory of Banach-Colmez spaces as developed in [**Col02**].

Definition 4.2.3. Let N be a filtered isocrystal over K. Let us write $\text{rk}(N)$ and $\deg(N)$ respectively for the rank and the degree of N as an isocrystal over K_0.

(1) We define the *degree* of the filtered vector space N_K, denoted by $\deg(N_K)$, to be the unique integer d with $\text{Fil}^d(\det(N_K)) \neq 0$.

(2) We define the *degree* of N by

$$\deg^\bullet(N) := \deg(N) - \deg(N_K).$$

(3) If N is not zero, we define its *slope* by

$$\mu^\bullet(N) := \frac{\deg^\bullet(N)}{\text{rk}(N)}.$$

Remark. It is straightforward to verify that MF_K^φ is a slope category as remarked after Theorem 3.3.22. Hence every $N \in \text{MF}_K^\varphi$ admits a unique Harder-Narasimhan filtration.

Example 4.2.4. Let V be a crystalline Γ_K-representation. We wish to show that $D_{\mathrm{cris}}(V)$ has degree 0. Proposition 3.2.14 from Chapter III implies that $\det(V)$ is a crystalline Γ_K-representation with $\det(D_{\mathrm{cris}}(V)) \cong D_{\mathrm{cris}}(\det(V))$, and consequently yield

$$\deg^\bullet(D_{\mathrm{cris}}(V)) = \deg^\bullet(\det(D_{\mathrm{cris}}(V))) = \deg^\bullet(D_{\mathrm{cris}}(\det(V))).$$

Hence we may replace V with $\det(V)$ to assume $\dim_{\mathbb{Q}_p} V = 1$.

Let us choose a continuous character $\eta : \Gamma_K \longrightarrow \mathbb{Q}_p^\times$ with $V \simeq \mathbb{Q}_p(\eta)$. Proposition 2.4.4 and Proposition 3.2.8 from Chapter III together imply that V is Hodge-Tate with

$$D_{\mathrm{cris}}(V)_K \cong D_{\mathrm{dR}}(V) \qquad \text{and} \qquad \mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V).$$

Hence Proposition 1.1.13 from Chapter III yields an integer n such that $\eta\chi^n(I_K)$ is finite. It follows by Theorem 1.1.8 from Chapter III that n is the Hodge-Tate weight of V, which in turn implies $\deg(D_{\mathrm{cris}}(V)_K) = n$.

It remains to show that $D_{\mathrm{cris}}(V)$ has degree n as an isocrystal. Let us denote by K^{un} the maximal unramified extension of K in $\overline{K}$, and by $\widehat{K^{\mathrm{un}}}$ the p-adic completion of K^{un}. We also write $W(\overline{k})$ for the ring of Witt vectors over $\overline{k}$, and $\widehat{K_0^{\mathrm{un}}}$ for the fraction field of $W(\overline{k})$. Example 3.2.2 and Proposition 3.2.13 from Chapter III together imply that $V(n) \simeq \mathbb{Q}_p(\eta\chi^n)$ is crystalline with

$$D_{\mathrm{cris}}(V(n)) \cong D_{\mathrm{cris}}(V) \otimes_K D_{\mathrm{cris}}(\mathbb{Q}_p(n)). \tag{4.6}$$

We then find by Example 3.2.9 from Chapter III that $\eta\chi^n(I_K)$ is trivial. Moreover, by construction $\widehat{K^{\mathrm{un}}}$ is a p-adic field with I_K as the absolute Galois group. Therefore we have

$$D_{\mathrm{cris}}(V(n)) = (V(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \subseteq (V(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{I_K} \cong B_{\mathrm{cris}}^{I_K} \cong \widehat{K_0^{\mathrm{un}}}$$

where the last identification follows from Theorem 3.1.8 from Chapter III. It follows by Proposition 3.2.7 from Chapter III that the Frobenius automorphism of $D_{\mathrm{cris}}(V(n))$ extends to the Frobenius automorphism of $\widehat{K_0^{\mathrm{un}}}$, which in turn implies that $D_{\mathrm{cris}}(V(n))$ has degree 0 as an isocrystal. In addition, as we have $\varphi(t) = pt$ by construction, we deduce by Example 3.2.2 from Chapter III that $D_{\mathrm{cris}}(\mathbb{Q}_p(n))$ has degree $-n$ as an isocrystal. The assertion is now straightforward to verify by the natural isomorphism (4.6) in MF_K^{φ}.

Definition 4.2.5. Let N be a filtered isocrystal over K.

(1) We say that N is *semistable* if we have $\mu^\bullet(M) \leq \mu^\bullet(N)$ for every nonzero filtered subisocrystal M of N.
(2) We say that N is *weakly admissible* if it is semistable of slope 0.
(3) We say that N is *admissible* if it is in the essential image of D_{cris}.

Proposition 4.2.6. *Every admissible filtered isocrystal over K is weakly admissible.*

Remark. The proof of Proposition 4.2.6 is mostly an elementary algebra, after replacing K by the completion of the maximal unramified extension of K in light of the remark after Proposition 3.2.20 from Chapter III. Curious readers can find a detailed proof in [**BC**, Theorem 9.3.4].

Proposition 4.2.7. *Let N be a weakly admissible filtered isocrystal over K, and set*

$$V := (N \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0(N_K \otimes_K B_{\mathrm{dR}}).$$

(1) V is naturally a crystalline Γ_K-representation with $\dim_{\mathbb{Q}_p}(V) \leq \dim_{K_0}(N)$.
(2) N is admissible if and only if we have $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(N)$.

Remark. We refer the readers to [**BC**, Proposition 9.3.9] for a complete proof. If N is admissible, the assertions are evident by Proposition 3.2.18 from Chapter III.

Proposition 4.2.8. *Let N be a filtered isocrystal over K.*

(1) There exists a unique vector bundle $\mathcal{F}(N)$ on X with

$$H^0(U, \mathcal{F}(N)) \cong (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \quad \textit{and} \quad \widehat{\mathcal{F}(N)_\infty} \cong \text{Fil}^0(N_K \otimes_K B_{\text{dR}})$$

where $\widehat{\mathcal{F}(N)_\infty}$ denotes the completed stalk of $\mathcal{F}(N)$ at ∞.

(2) We have $\text{rk}(N) = \text{rk}(\mathcal{F}(N))$, $\deg^\bullet(N) = \deg(\mathcal{F}(N))$ and $\mu^\bullet(N) = \mu(\mathcal{F}(N))$.

(3) N is weakly admissible if and only if $\mathcal{F}(N)$ is semistable of slope 0.

Remark. A complete proof of Proposition 4.2.8 may be added later. Here we explain some key ideas as sketched in [**FF18**, Lemma 10.5.5 and Proposition 10.5.6].

The first statement follows from Proposition 4.2.1 once we verify verify using Theorem 2.3.15 from Chapter II that $(N \otimes_{K_0} B_{\text{cris}})^{\varphi=1}$ is a free B_e-module with an identification

$$(N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \otimes_{B_e} B_{\text{dR}} \cong N_K \otimes_K B_{\text{dR}}.$$

The second statement can be obtained by realizing $\mathcal{F}(N)$ in a short exact sequence

$$0 \longrightarrow \mathcal{F}(N) \longrightarrow \mathcal{E}(N) \longrightarrow \mathcal{T} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf supposed at ∞. The third statement is obtained as a special case of the fact that the functor $\mathcal{F}$ preserves the Harder-Narasimhan filtration, which is not hard to prove by observing that the Harder-Narasimhan filtrations of N and $\mathcal{F}(N)$ are stable under the natural actions of Γ_K.

Theorem 4.2.9 (Colmez-Fontaine [**CF00**])**.** *A filtered isocrystal N over K is admissible if and only if it is weakly admissible.*

Proof. If N is admissible, then it is weakly admissible by Proposition 4.2.6. Let us now assume that N is weakly admissible, and set

$$V := (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(N_K \otimes_K B_{\text{dR}}).$$

In light of Proposition 4.2.7, it suffices to show $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(N)$. Proposition 4.2.8 yields a semistable vector bundle $\mathcal{F}(N)$ on X of slope 0 with

$$H^0(U, \mathcal{F}(N)) \cong (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \quad \text{and} \quad \widehat{\mathcal{F}(N)_\infty} \cong \text{Fil}^0(N_K \otimes_K B_{\text{dR}})$$

where $\widehat{\mathcal{F}(N)_\infty}$ denotes the completed stalk of $\mathcal{F}(N)$ at ∞. Hence by Proposition 4.2.1 we obtain a canonical isomorphism

$$H^0(X, \mathcal{F}(N)) \cong (N \otimes_{K_0} B_{\text{cris}})^{\varphi=1} \cap \text{Fil}^0(N_K \otimes_K B_{\text{dR}}) = V.$$

Moreover, Theorem 3.5.3 and Proposition 4.2.8 together imply that $\mathcal{F}(N)$ is isomorphic to $\mathcal{O}_X^{\oplus r}$ where we set $r := \dim_{K_0}(N)$, and consequently yields an isomorphism

$$V \cong H^0(X, \mathcal{F}(N)) \simeq H^0(X, \mathcal{O}_X)^{\oplus r} \cong \mathbb{Q}_p^{\oplus r}$$

by Proposition 3.1.6 and Theorem 3.2.9. We thus find $\dim_{\mathbb{Q}_p}(V) = \dim_{K_0}(N)$ as desired. □

Remark. While the proof above greatly simplifies the original proof by Colmez-Fontaine [**CF00**] and another proof by Berger [**Ber08**], these prior proofs contained a number of important ideas that contributed to the discovery of the Fargues-Fontaine curve.

Corollary 4.2.10. *The functor D_{cris} is an equivalence between $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(\Gamma_K)$ and the category of weakly admissible filtered isocrystals over K.*

Proof. This is immediate by Theorem 3.2.19 from Chapter III and Theorem 4.2.9. □

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